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A THEOREM ON INVARIANT ANALYTIC FUNCTIONS WITH APPLICATIONS TO RELATIVISTIC QUANTUM FIELD THEORY

BY

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Synopsis.

The paper is in three parts of which the first two are mathematical. In the first part, a detailed proof is given of a previously announced theorem: an analytic function of n four vector variables invariant under the orthochronous Lorentz group is an analytic function of their scalar products. The second part is devoted to a preliminary study of the domain of analyticity of such invariant analytic functions. The third part applies the preceding results to quantum field theory. It is shown that the vacuum expectation value $(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0) = F^{(n)}(x_1, \ldots x_n)$ where $\varphi(x)$ is neutral scalar field, is an analytic function of the real variables $x_j - x_{j+1}$, $j = 1, \ldots n-1$ in a region where all these vectors are space-like. It is shown that the values of $F^{(n)}$ for all values of its arguments are uniquely determined in terms of its values for space-like separations, and that, for $n = 2, 3, 4$, $F^{(n)}$ is determined from its values at points where all times are equal. These results are applied to prove generalizations of two theorems of R. Haag. In effect, these theorems show that, to give different physical predictions, two theories of an interacting field which satisfies the canonical commutation relations must use inequivalent representations of the commutation relations.

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Introduction.

In a preceding paper[1], the second-named author showed that the main content of a relativistic quantum theory of a scalar field, $\varphi(x)$, is contained in the vacuum expectation values, $F^{(n)}$, defined by

$$F^{(n)}(x_1, \dots x_n) = (\Psi_0, \varphi(x_1) \dots \varphi(x_n) \Psi_0), \; n = 1, 2, \dots,$$

where Ψ_0 is the vacuum state. It was shown there that, as a consequence of the transformation law of the field under space-time translations and the absence of negative energy states, the distributions $F^{(n)}$ are boundary values of analytic functions. The analysis of the structure of the $F^{(n)}$ was carried further, using a theorem, quoted there without proof, which may be stated roughly as follows: an analytic function of n four-vector variables invariant under the orthrochronous Lorentz group is an analytic function of their scalar products.

The first part of the present paper is devoted to a proof of this theorem. Because the techniques introduced in the proof have further useful applications in quantum field theory, we have given a detailed exposition.

The second part of the paper contains a preliminary study of the set, $\mathfrak{M}_n$, of symmetric $n \times n$ complex matrices, Z, defined by $Z_{jk} = z_j \cdot z_k$, $j, k = 1, \dots n$, where $z_1, \dots z_n$ are complex four vectors of the form $z_j = \xi_j - i\eta_j$, with ξ_j and η_j real and η_j in the interior of the future light cone (this set of $z_1, \dots z_n$ is called the tube). According to the theorem proved in the first part of the paper, the set $\mathfrak{M}_n$ is a domain of analyticity of the invariant analytic function which has the physical $F^{(n)}$ as its boundary value when all $\eta_j \to 0$. It is shown that there are points, $z_1, \dots z_n$, on the boundary of the tube which yield matrices, $z_i \cdot z_j$, of scalar products lying in the interior of $\mathfrak{M}_n$. From this simple geometrical fact, it follows that an invariant function analytic in the tube cannot have an arbitrary invariant distribution as boundary value. In fact, it turns out that the boundary value has to be an analytic function of the real variables $\xi_j \cdot \xi_k$, $j, k = 1, \dots n$ in a certain domain and that the analytic function is uniquely determined once its values are known in certain subdomains of the boundary of the tube.

In the third part of the paper, the results of the preceding sections are applied to the vacuum expectation values of a scalar field. It is shown that $F^{(n)}(x_1, \ldots x_n)$ is an analytic function of the real variables $(x_j - x_{j+1}) \cdot (x_k - x_{k+1})$, $j, k = 1, \ldots n-1$ when all $x_j - x_{j+1}$ are space-like and lie in a certain region. It is further shown that the values of $F^{(n)}$ for all values of its arguments are uniquely determined in terms of its values for space-like separation. For the cases $n = 2, 3, 4$, an even stronger result is obtained: $F^{(n)}$ is determined everywhere from its values at points where all the times (x_{j0}), $j = 1. \ldots n$ are equal. These results are applied to prove generalizations of two theorems of R. Haag, which can be stated roughly as follows: First, let there be given two theories of a field which transforms as a scalar under the rotations and translations of three space at a fixed time. Suppose that the canonical variables of the theories are unitary equivalent at that time via a unitary transformation V. Then, the representations of the Euclidean group of the two theories are unitary equivalent via V. Second, if the two theories just described are, in addition, invariant under the inhomogeneous Lorentz group and have no negative energy states and unique vacuum states, then the vacuum expectation values $(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0)$ are identical in the two theories for $n = 1, 2, 3, 4$. The paper closes with a discussion of the physical significance of this generalized Haag's theorem.

We want to emphasize that the main results of the paper, as far as the structure of the $F^{(n)}$ are concerned (its determination everywhere in terms of its values for space-like separated arguments), are valid in both local and non-local field theory.

1. An Invariant Analytic Function of Vectors is an Analytic Function of Scalar Products.

The following theorem was stated without proof in I.

Theorem 1.

Let f be a complex valued function of n four-vector variables, $z_j = \xi_j - i\eta_j$, $j = 1, \ldots n$, where ξ_j and η_j are real. Suppose f is analytic in the tube defined by

$$\infty < \xi_{j\mu} < \infty, \; j = 1, \ldots \; n, \; \mu = 0, 1, 2, 3;$$

$$\eta_j \text{ in the future cone, i. e. } \eta_j^2 > 0, \; \eta_{j0} > 0, \; j = 1, \ldots n$$

and invariant under the orthochronous homogeneous Lorentz group, $L^{\uparrow}$:

$$f(z_1, \ldots z_n) = f(\Lambda z_1, \ldots \Lambda z_n) \text{ for } \Lambda \, \varepsilon \, L^{\uparrow}. \tag{1}$$

Then, f is a function of the scalar products $z_j \cdot z_k$, $j, k = 1, \ldots n$. It is analytic on the complex variety, $\mathfrak{M}_n$, over which the scalar products vary when the vectors $z_1, \ldots z_n$ vary over the tube.

Outline of the Proof.

If a function is analytic in the tube and satisfies (1) for $\Lambda \, \varepsilon \, L^{\uparrow}$, then (Lemma 1) it also satisfies (1) when Λ is an arbitrary complex Lorentz transformation, i. e., a matrix $\Lambda^{\mu}{}_{\nu}$ $\mu, \nu = 0, 1, 2, 3$, whose elements are complex numbers satisfying $\Lambda^{\mathsf{T}} \Lambda = 1$, which means

$$\sum_{\mu=0}^{3} \Lambda_{\mu}{}^{\varkappa} \Lambda^{\mu}{}_{\nu}, = g^{\varkappa}{}_{\nu}. \tag{2}$$

We call the set of such matrices the *complex Lorentz group*, $\mathfrak{L}$. Furthermore, if $\Lambda z_1, \ldots \Lambda z_n$ lies outside the future tube, (1) defines a single valued analytic continuation of the function f originally given. We shall refer to the set of points $\Lambda z_1, \ldots \Lambda z_n$, for $\Lambda \, \varepsilon \, \mathfrak{L}$ and $z_1, \ldots z_n$ in the tube, as the *extended tube*.

Next, (Lemma 2), we examine the sets of n vectors $z_1, \ldots z_n$ and $\zeta_1, \ldots \zeta_n$ which satisfy

$$z_j \cdot z_k = \zeta_j \cdot \zeta_k \quad j, k = 1 \ldots n. \tag{3}$$

If the $n \times n$ matrix of the scalar products has rank three or four, there exists a complex Lorentz transformation Λ such that

$$\Lambda z_i = \zeta_i \quad i = 1, \ldots n.$$

If the rank is two or one, the connection between z_i and ζ_i is more complicated:

$$\Lambda z_i = \zeta_i + \alpha_i \omega,$$

where the α_i, $i = 1, \ldots n$ are complex numbers and ω is a vector of zero length orthogonal to Λz_i and ζ_i, $i = 1, \ldots n$. For points at which the rank is three or four, the invariance of an analytic function, f, of n vectors $z_1, \ldots z_n$, clearly implies the single valuedness of f regarded as a function of the scalar products (3). For points where the rank is two or one, a further argument is necessary and is supplied. Thus, f is a single valued function everywhere on the variety, $\mathfrak{M}_n$ defined by the scalar products $z_j \cdot z_k$, $j, k = 1, \ldots n$. The points of $\mathfrak{M}_n$ are labeled by the $1/2\ n(n+1)$ scalar products. $\mathfrak{M}_n$ can be regarded as an algebraic variety in the space of all complex $n \times n$ symmetric matrices. In fact, it turns out to be an open subset of the set of all complex symmetric matrices of rank $\leqslant 4$ and, as such, has dimension $1\ (n=1)$, $3\ (n=2)$, $4\,n-6\ (n \geqslant 3)$.

In order to be able to connect the continuity properties of invariant functions of vectors with their corresponding properties regarded as functions of scalar products, it is necessary to investigate the connection between neighbourhoods of sets of vectors and neighbourhoods of their sets of scalar products (Lemma 3). This connection is quite a simple one at points of $\mathfrak{M}_n$ where the rank is three or four, but where it is two or one the situation is quite delicate, because the structure of the set of points in the space of the vectors which map into a given point of $\mathfrak{M}_n$ is essentially more complicated. Nevertheless, the proof of the continuity of f as a function on $\mathfrak{M}_n$ can be, and is, carried out.

To complete the proof of the theorem, it remains to show that f is analytic on $\mathfrak{M}_n$. For $n \leqslant 4$, analyticity is a perfectly straightforward notion because $\mathfrak{M}_n$ is an open set in complex Euclidean $1/2\ n\,(n+1)$ space. However, for $n \geqslant 5$, $\mathfrak{M}_n$ is an open set on a $4\,n-6$ dimensional algebraic variety and the notion of analyticity requires some explanation. For a point P of $\mathfrak{M}_n$ ($\geqslant 5$) for which the rank is four, the tangent space to $\mathfrak{M}_n$ at P has dimension $4\,n-6$. (Recall that the tangent space to $\mathfrak{M}_n$ at P is the linear manifold of the space of all complex symmetric matrices spanned by the tangent vectors to $\mathfrak{M}_n$ at P.) Sufficiently small neighbourhoods of P on $\mathfrak{M}_n$ can be put in analytic one to one correspondence with sufficiently small neighbourhoods in the tangent space. Near such points, f can be regarded as defined in a neighbourhood of the origin in a complex $(4\,n-6)$-dimensional Euclidean space and its analyticity defined in the well-known way. The points of $\mathfrak{M}_n$, $n\geqslant 5$, where the rank is less than four are *singular* in the terminology of algebraic geometry 2*. For them, the tangent vector space has dimension $1/2\;n(n+1)$ and neighbourhoods are not locally Euclidean. (The reader may find it helpful to think of the example of the light cone. In that case, the point where the tips of the past and future cones touch is singular and its neighbourhoods are not locally Euclidean. However, it should be borne in mind that the actual situation is much more complicated since singular points only appear on $\mathfrak{M}_n$ for $n\geqslant 5$ and in the simplest case, $n=5$, already form a variety of 24 (real) dimensions.) Evidently, the above definition of analyticity does not apply at such a point. It is not impossible to extend the notion of analyticity to apply there. In fact, one can do it in a number of different ways. However, it can happen that physically important consequences of ordinary analyticity do not hold for "generalized analyticity". In the following, we prove analyticity at all points of $\mathfrak{M}_n$ for $n\leqslant 4$, analyticity at non-singular points for $n\geqslant 5$, and boundedness and continuity at singular points for $n\geqslant 5$, and that is what is to be understood by "analytic on $\mathfrak{M}_n$" in the statement of the theorem. It actually is sufficient to guarantee analyticity in the sense of Bochner and Martin[3].

The proof of the analyticity is completed in four steps. First, differential equations are derived which f satisfies by virtue of its invariance under $L^{\uparrow}$ or $\mathfrak{L}$ (Lemma 4). The scalar products $z_i \cdot z_j$, $j=1, \ldots n$, considered as functions of the vectors $z_1, \ldots z_n$, satisfy these differential equations. Next, it is shown that, in a neighbourhood of a point of $\mathfrak{M}_n$ at which the scalar products span all solutions of the differential equations, f is expandable in a power series of appropriately chosen scalar products (Lemma 5). Lemmas 6 and 7 then show that at every non-exceptional point the scalar products satisfy the conditions of Lemma 5. Finally, to complete the proof, a theorem

* For $n\leqslant 4$, we shall refer to the points P of $\mathfrak{M}_n$ at which the rank is less than maximum as exceptional although, for $n\leqslant 4$, they are not singular in the sense of algebraic geometry.

on removable singularities is used to show that f is analytic even at the exceptional points for $n \leqslant 4$.

Lemma 1.

Let $f(z_1, \ldots z_n)$ be analytic in the tube and invariant under the orthochronous real Lorentz group, $L^{\uparrow}$. Then, f is also invariant under the complex Lorentz group $\mathfrak{L}$, as long as $\Lambda z_1, \ldots \Lambda z_n$ is in the tube. When $\Lambda z_1, \ldots \Lambda z_n$ lies out of the tube, the relation $f(\Lambda z_1, \ldots \Lambda z_n) = f(z_1 \ldots z_n)$ defines a single valued analytic continuation of f to the extended tube.

Proof.

Let $z_1, \ldots z_n$ be a fixed point of the tube. Then, for all Λ in a suitable neighbourhood of the identity in $\mathfrak{L}$, $\Lambda z_1, \ldots \Lambda z_n$ again lies in the tube. In some sub-neighbourhood, N, we can introduce canonical coordinates $\lambda_1, \ldots \lambda_6$ such that[4]

1. As Λ runs over N, $\lambda_1, \ldots \lambda_6$ vary over a neighbourhood N' of the origin in the complex six-dimensional Euclidean space with (complex) coordinates $\lambda_1, \ldots \lambda_6$.

2. The subset of N, for which $\Lambda \varepsilon L^{\uparrow}$, is the subset of N' for which the $\lambda_1, \ldots \lambda_6$ are real.

3. The matrix elements $\Lambda^{\mu}{}_{\nu}$ (and therefore the vector components $\sum\limits_{\nu=0} \Lambda^{\mu}{}_{\nu}\, z^{\nu}$) are analytic functions of $\lambda_1, \ldots \lambda_6$.

Since an analytic function of analytic functions is again analytic, $f(\Lambda z_1, \ldots \Lambda z_n)$ is an analytic function of $\lambda_1, \ldots \lambda_6$ in N'. Furthermore it has the property that for *real* $\lambda_1, \ldots \lambda_6$ it is constant. Therefore it is also constant for complex $\lambda_1, \ldots \lambda_6$ in N'[5].

Thus, for $\Lambda \varepsilon N \subset \mathfrak{L}$, equation (1) is satisfied.

This result can be extended immediately in two ways. First, the argument applies when Λ runs over the neighbourhood $I_s N$ of the space inversion, $\Lambda = I_s$. Second, (1) also holds if $\Lambda z_1, \ldots \Lambda z_n$ can be connected to $z_1, \ldots z_n$ by a curve

$$\Lambda(t) z_1, \ldots \Lambda(t) z_n;\ 0 \leqslant t \leqslant 1;\ \Lambda(0) = 1;\ \Lambda(1) = \Lambda,$$

lying entirely within the tube and such that it can be covered by a finite number of overlapping neighbourhoods: $\Lambda(t_j) N z_1, \ldots \Lambda(t_j) N z_n$, *lying within the tube.*

However, this last argument by no means completes the proof of (1), because it is not clear that all pairs of points $z_1, \ldots z_n$ and $\Lambda z_1, \ldots \Lambda z_n$, each of which is in the tube, can be connected by a curve of the sort described above. (If Λ is improper, it is $I_s z_1, \ldots I_s z_n$ and $\Lambda z_1, \ldots \Lambda z_n$, which have to be connected by the curve. For simplicity of statement, we consider only proper Λ in the rest of this proof. The extension to improper Λ is trivial). We shall give an explicit construction of such a curve at the end of the proof of this Lemma. Assuming the construction for the present, we have completed the proof of the first statement of the Lemma.

The existence of curves of the above described type is closely connected with the possibility of making a single valued analytic continuation of $f(z_1, \ldots z_n)$ to the extended tube. For, starting from a fixed point $z_1, \ldots . z_n$ of the tube, we can extend the analytic function $f(\Lambda z_1, \ldots \Lambda z_n)$ of Λ over the whole complex Lorentz group, $\mathfrak{L}$. (It is the simplest possible analytic function on $\mathfrak{L}$, a constant.) f is then defined for points $\Lambda z_1, \ldots \Lambda z_n$ of the extended tube. Starting from a different point $z'_1 \ldots . z'_n$ of the tube, f can be defined for the points $Mz'_1, \ldots Mz'_n$, $M \varepsilon \mathfrak{L}$. If it happens that for some Λ and M, $\Lambda z_j = Mz'_j$, $j = 1, \ldots n$, the single valuedness of the extension of f would be insured by: $f(z_1, \ldots z_n) = f(z'_1, \ldots z'_n) = f(M^{-1}\Lambda z_1, \ldots M^{-1}\Lambda z_n)$. It is just this identity which is guaranteed by our postponed construction of curves, and therefore f as extended is single valued.

The analyticity of f in the extended tube at $\Lambda z_1, \ldots \Lambda z_n$ follows from its analyticity at $z_1, \ldots z_n$ in the tube, because the partial derivatives at $\Lambda z_1, \ldots \Lambda z_n$ are expressible in terms of partial derivatives at $z_1, \ldots . z_n$, e.g.,

$$\frac{\partial f}{\partial (\Lambda z_1)_\mu}(\Lambda z_1, \ldots \Lambda z_n) = \sum_{\nu=0}^{3} \frac{\partial f(z_1, \ldots z_n)}{\partial (z_1)^\nu} \frac{\partial (z_1)^\nu}{\partial (\Lambda z_1)_\mu}.$$

This completes the proof of the second statement of the Lemma.

It remains to construct a curve $\Lambda(t)z_1, \ldots \Lambda(t)z_n$; $0 \leqslant t \leqslant 1$, $\Lambda(t) \varepsilon \mathfrak{L}$, beginning at an arbitrary point of the tube $z_1, \ldots z_n$, ending at the point $\Lambda z_1, \ldots \Lambda z_n$ of the tube, and lying entirely within the tube. The existence of such a curve is obvious if Λ is a *real* (orthochronous, proper) Lorentz transformation, because every such transformation leaves the tube invariant and their set is connected.

For Λ complex, the required calculations are simpler in a two dimensional matrix formalism in which the four vector z^μ is represented by the matrix

$$Z = \begin{pmatrix} z^0 + z^3 & z^1 - iz^2 \\ z^1 + iz^2 & z^0 - z^3 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}. \tag{4}$$

Then the most general proper complex Lorentz transformation is of the form $z^\mu \to z'^\mu$, where the four vector z'^μ belongs to a matrix, Z', given by

$$Z' = AZB^*. \tag{5}$$

Here A and B are 2×2 matrices of determinant one. In particular, the most general (orthochronous, proper) real Lorentz transformation is of this form with $A = B$. This last fact permits us to simplify our problem. Note that (5) can be written

$$Z' = (AB^{-1}) BZB^*, \tag{6}$$

so that the most general complex Lorentz transformation is of the form of a real Lorentz transformation followed by a complex Lorentz transformation of the special form

$$Z' = CZ. \tag{7}$$

It therefore suffices to consider complex Lorentz transformations of this special form. The problem can be simplified further by making a suitable real Lorentz transformation of the final vectors

$$Z' \to DZ'D^* = (DCD^{-1})\, DZD^*.$$

By suitable choice of D, we can bring C into triangular form. In fact, unless the proper values of C are equal, C can be diagonalized. Thus, we can restrict our attention to C of the form

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \tag{8}$$

and

$$\begin{pmatrix} \pm 1 & \tau \\ 0 & \pm 1 \end{pmatrix}. \tag{9}$$

We shall prove that, if Z is in the tube and $Z' = CZ$ is in the tube where C is of the form (8) or (9), then $C(t)Z$ is in the tube where

$$C(t) = \begin{pmatrix} \exp[t(\varrho + i\theta)] & 0 \\ 0 & \exp[-t(\varrho + i\theta)] \end{pmatrix}, \quad \left.\begin{matrix} 0 \leqslant t \leqslant 1, \\ \zeta = \exp(\varrho + i\theta), \end{matrix}\right\} \tag{10}$$

$$C(t) = \begin{pmatrix} \pm 1 & t\tau \\ 0 & \pm 1 \end{pmatrix}, \quad 0 \leqslant t \leqslant 1, \tag{11}$$

respectively. Thus, it will be possible to choose the same curve independent of the point Z. (Actually, it will be seen that the case of the minus sign in (11) can be excluded.)

The conditions which express in terms of the matrix Z' the fact that the vector $z'_\mu = \xi'_\mu - i\eta'_\mu$ lies in the tube are

$$-4(\eta')^2 = \det(Z' - Z'^*) < 0, \tag{12}$$

$$\eta'_0 = -\frac{1}{2}\,\mathrm{Im}\; tr(Z') > 0, \tag{13}$$

as can be derived by a simple computation. Our procedure will be to determine all ζ and τ in (8) and (9) consistent with (12) and (13) for fixed z^μ in the tube. We shall see that, if ζ and τ are consistent, so are ζ^t and $t\tau$ for $0 \leqslant t \leqslant 1$.

Consider first the case (9). Then, the condition (12) reads

$$0 > -4(\eta')^2 = \det\begin{pmatrix} \pm(Z_{11} - \overline{Z}_{11}) + \tau Z_{21} - \overline{\tau} Z_{21}, & \pm(Z_{12} - \overline{Z}_{21}) + \tau Z_{22} \\ \pm(Z_{21} - \overline{Z}_{12}) - \overline{\tau}\,\overline{Z}_{22}, & \pm(Z_{22} - \overline{Z}_{22}) \end{pmatrix}$$

$$= |\tau|^2 |Z_{22}|^2 \mp (s\overline{\tau} + \overline{s}\tau) + \det(Z - Z^*),$$

$$\text{where} \quad s = \overline{Z}_{21} Z_{22} - \overline{Z}_{22} Z_{12}.$$

This inequality can be rewritten in the form

$$|\tau \mp |Z_{22}|^{-2} s|^2 - [|Z_{22}|^{-4}|s|^2 + |Z_{22}|^{-2}(-\det(Z - Z^*))] < 0$$

which describes the interior of a circle in the τ plane about the point

$$\pm |Z_{22}|^{-2} s.$$

As far as the condition (13) is concerned, we note that it is satisfied by all points in the interior of the circle if it is satisfied by any one, because the vector η' must pass through a vector of zero length in order to change the sign of η'_0. With the plus sign in (9), the condition (13) is always satisfied and never for the minus sign, as one sees by considering the case $\tau = 0$.

Evidently, if the point τ is in the interior of the allowed circle, the points $t\tau$, $0 \leqslant t \leqslant 1$ will also be because the origin is in the circle and the circle is convex.

Now we turn to the case (8) in which C is diagonal. Here, there are also two two-dimensional domains of C's consistent with condition (12), one of which is excluded by (13), as we shall see by a detailed consideration. The boundary of the allowed domain is convex in terms of the parameters ϱ and θ defined in (10). This will have to be proved by a detailed computation since the boundary is not an elementary curve.

Condition (12) for the case (8) is

$$0 > -4(\eta')^2 = |Z_{12}|^2 |\zeta|^2 + |Z_{21}|^2 |\zeta|^{-2}$$
$$-(Z_{11}\overline{Z}_{22}\zeta\overline{\zeta}^{-1} + \overline{Z}_{11} Z_{22}\overline{\zeta}\zeta^{-1}) + 2\,Re\,(\det Z). \tag{14}$$

This condition is satisfied for $-\zeta$ if it is satisfied for ζ and, in particular, since by assumption it is satisfied for $\zeta = +1$, it is also satisfied for $\zeta = -1$. For $\zeta = -1$, the vector z' is not in the tube. Consequently, by the same type of continuity argument which we used in connection with equation (9), there must be at least two disconnected sets of ζ satisfying (14), one of which does not satisfy (13). Another consequence of (13), which we shall use in the following, is $Z_{11}\bar{Z}_{22} \neq 0$.

We divide the remainder of the argument into three parts.

Case 1.

$$Z_{12} \neq 0, \; Z_{21} \neq 0.$$

In this case, we can divide (14) by $2|Z_{12}|\,|Z_{21}|$, and introduce the new variables

$$\left.\begin{aligned} &\sigma_0 = \arg(\bar{Z}_{11} Z_{22}), \; \zeta = \exp(\varrho + i\theta), \\ &\chi = 2\varrho - ln\,(|Z_{21}||Z_{12}|^{-1}), \; \psi = 2\theta - \sigma_0, \\ &\nu = -|Z_{21}|^{-1}|Z_{12}|^{-1}\,Re\,(\det Z), \; \mu = |Z_{11}||Z_{22}||Z_{12}|^{-1}|Z_{21}|^{-1} > 0. \end{aligned}\right\} \tag{15}$$

Equation (14) then reads

$$0 > P = \cosh\chi - (\mu\cos\psi + \nu). \tag{16}$$

Since P is periodic in ψ, it suffices to consider (16) in the strip $|\psi| \leq \pi$, $-\infty < \chi < \infty$, and show that it defines a convex region there.

There are two conditions on the coefficients μ and ν:

$$|\nu| \leqslant 1 + \mu, \tag{17}$$

and

$$\nu + \mu > 1. \tag{18}$$

The first of these merely says the real part of the determinant of a matrix is less in absolute value than the sum of the absolute values of the terms which comprise the determinant. The second is a consequence of the fact that for $\zeta = 1$, $P < 0$ by assumption, and therefore $\nu + \mu\cos\sigma_0 > 1$.

From (17) and (18) it follows that there exists an angle ψ_0, $0 < \psi_0 \leqslant \pi$, such that

$$\nu + \mu\cos\psi_0 = 1.$$

For

$$|\psi| \leqslant |\psi_0|, \; \nu + \mu\cos\psi \geqslant 1. \tag{19}$$

The expression (16) for P makes it clear that its behaviour in χ for fixed ψ is: if $\mu\cos\psi + \nu < 1$, $P > 0$ for all χ; if $\mu\cos\psi + \nu = 1$, $P = 0$ for $\chi = 0$ and $P > 0$

for $\chi \neq 0$; if $\mu \cos \psi + \nu > 1$, $P = 0$ for some $\chi_1 = \chi_1(\psi) > 0$ (and also for $\chi = -\chi_1$), $P < 0$ for $|\chi| < \chi_1$, and $P > 0$ for $|\chi| > \chi_1$.

The discussion of the preceding two paragraphs shows that the set of χ and ψ satisfying $P < 0$ in the strip $|\psi| \leqslant \pi$, $-\infty < \chi < \infty$, is connected and invariant under reflections in the $\chi = 0$ and $\psi = 0$ axes, being the domain: $-\chi_1(\psi) < \chi < \chi_1(\psi)$, $-\psi_0 < \psi < \psi_0$, where χ_1 is given by

$$\cosh \chi_1 = \mu \cos \psi + \nu, \; \chi_1 > 0. \tag{20}$$

To complete the argument we will show the convexity of the function χ_1 as a function of ψ.

Differentiating (20) twice and eliminating the first derivative, χ_1', of χ with respect to ψ, we find, for $|\psi| < \psi_0$, $(\sinh \chi_1)^3 \chi_1'' = -\mu Q$, where

$$Q = \cos \psi (\sinh \chi_1)^2 + \mu (\sin \psi)^2 \cosh \chi_1 \tag{21}$$

$$= (\mu \cos \psi + \nu)(\mu + \nu \cos \psi) - \cos \psi, \tag{22}$$

or

$$Q = \mu\nu (\cos \psi + 1)^2 - [1 - (\mu - \nu)^2] \cos \psi. \tag{23}$$

We assert that $Q > 0$ for $|\psi| < \psi_0$. For $0 \leqslant \psi \leqslant \frac{\pi}{2}$, this is an immediate consequence of (21), since both terms on the right hand side of (21) are positive there. For $1/2\, \pi < \psi < \psi_0$, we have $\cos \psi_0 < 0$, and from (19), $\nu = 1 + \mu |\cos \psi_0|$. Using this last fact, we see that, when $\mu > \nu$, $\mu \cos \psi + \nu > 0$, $\mu + \nu \cos \psi > 0$ and $-\cos \psi > 0$ so that (22) immediately implies $Q > 0$. Again, when $\mu \leqslant \nu$, we use the form (23), and note that $1 - (\mu - \nu)^2 = (1 - \mu + \nu)(1 + \mu - \nu)$, and $1 + \nu - \mu > 0$, and $1 + \mu - \nu = \mu (1 - |\cos \psi_0|) \geqslant 0$, so $Q > 0$ in this case too.

Case II.

$$Z_{12} = 0, \; Z_{21} \neq 0 \; (\text{or } Z_{12} \neq 0, \; Z_{21} = 0).$$

Here we define σ_0, ϱ, θ, and ψ as in (15), but μ, ν, χ as follows

$$\chi = -2\varrho + ln\left(\frac{1}{2}|Z_{21}|^2\right), \; \mu = |Z_{11}||Z_{22}| > 0, \; \nu = -Re(\det Z). \tag{24}$$

Then, the basic inequality (14) takes the form

$$0 > P = e^{\chi} - (\mu \cos \psi + \nu), \tag{25}$$

and the analogues of the inequalities (17) and (18) are

$$|\nu| \leqslant \mu \tag{26}$$

and

$$\nu + \mu > 0 . \tag{27}$$

(If $Z_{12} \neq 0$, $Z_{21} = 0$, then, in equation (24), replace the definition of χ by $\chi = 2\,\varrho + ln\left(\frac{|Z_{12}|^2}{2}\right)$. This case is essentially the same as that for $Z_{12} = 0$ and $Z_{21} \neq 0$ and will not be discussed further.)

A discussion analogous to that in Case I shows that the region of the strip $|\psi| \leqslant \pi$, $-\infty < \chi < \infty$, as determined by $P < 0$, is given by

$$-\infty < \chi < \chi_1(\psi), \quad |\psi| < \psi_0 \leqslant \pi,$$

$$\mu \cos \psi_0 + \nu = 0, \text{ and } e^{\chi_1} = \mu \cos \psi + \nu.$$

Here, $\exp(2\,\chi_1)\,\chi_1'' = -\mu Q$, where $Q = \mu + \nu \cos \psi > 0$, so that the region is convex.

Case III.

$$Z_{12} = Z_{21} = 0 .$$

Here the basic inequality is

$$0 > P = -(\mu \cos \psi + \nu),$$

with μ, ψ and ν defined as in Case II. μ and ν satisfy the same inequalities as in Case II, namely (26) and (27), so that there again exists a ψ_0 satisfying $\mu \cos \psi_0 + \nu = 0$, $0 < \psi_0 \leqslant \pi$, and the region permitted by $P < 0$ is the strip $|\psi| < \psi_0$, $-\infty < \chi < \infty$ which is obviously convex.

In each of these three cases we have proved that the region of χ, ψ space (or what is essentially the same thing since it is obtained by a translation and change of scale, ϱ, θ space) permitted by conditions (12) and (13) is convex. Since these regions contain the point $\varrho = 0$, $\theta = 0$, they also contain the points $t\varrho$, $t\theta$, $0 \leqslant t \leqslant 1$, corresponding to the transformations (10) and (11); so the proof of Lemma 1 is complete.

Lemma 2.

Let $z_1, \ldots z_n$ and $\zeta_1, \ldots \zeta_n$ be any two sets of n vectors such that

$$Z_{ij} = z_i \cdot z_j = \zeta_i \cdot \zeta_j, \; i, j = 1, \ldots n. \tag{28}$$

If the rank of the $n \times n$ matrix Z is three or four (or, for $n \leqslant 2$ if Z is non-singular), then there exists a complex Lorentz transformation, Λ, such that

$$\Lambda z_i = \zeta_i,\ i = 1, \dots n. \tag{29}$$

If the rank of Z is two or one (and $n > 2$ or $n > 1$, respectively), a $\Lambda \varepsilon\, \mathfrak{L}$ satisfying (29) will not exist, in general, but there always exists a Λ satisfying

$$\Lambda z_i = \zeta_i + \alpha_i \omega, \tag{30}$$

where α_i are complex numbers and ω is a vector of zero length orthogonal to ζ_i and Λz_i, $i = 1, \dots n$.

Proof.

We note the known fact that, for a symmetric matrix, the rank determined from principal minors is the same as the rank determined from all minors[6]. Thus, if the rank of Z is r, there exist r vectors, say $z_1, \dots z_r$, which have non-vanishing Gram determinant

$$0 \neq G(z_1, \dots z_r) = \det (z_i \cdot z_j),\ i, j = 1, \dots r.$$

This result will be used tacitly many times in the following.

The first step in the proof is to establish the connection between the condition, $G(z_1 \dots z_r) \neq 0$, and the linear independence of the set of vectors $z_1 \dots z_r$. If $G(z_1 \dots z_r) \neq 0$, then the set $z_1 \dots z_r$ is linearly independent. For a relation

$$\sum_{j=1}^{r} \alpha_j z_j = 0 \tag{31}$$

would imply

$$\sum_{j=1}^{r} \alpha_j (z_k \cdot z_j) = 0 \qquad k = 1 \dots r \tag{32}$$

and these last equations have a non-trivial solution $\alpha_1 \dots \alpha_r$ if and only if $\det (z_j \cdot z_k) = 0$, $j, k = 1 \dots r$. The converse, that the linear independence of the set $z_1 \dots z_r$ implies $G(z_1 \dots z_r) \neq 0$, is not true in general. For example,

$$z_1 = (0, 1, 0, 0), \quad z_2 = (1, 1, 1, 0)$$

is a pair of linearly independent vectors having zero Gram determinant. However, for $r = 4$, the converse holds, for $G(z_1 \dots z_4) = 0$ implies that the equations (32) have a non-trivial solution $\alpha_1, a_2, \alpha_3, \alpha_4$. This, in turn, implies that there is a vector of the form $\sum_{j=1}^{4} \alpha_i z_j$, with at least one $\alpha_j \neq 0$, which is orthogonal to all z_j. If the z_j were linearly independent, this last would be impossible, since $\sum_{j=1}^{4} \alpha_j z_j$ would then be orthogonal to every vector. Thus, $G(z_1 \dots z_4) = 0$ implies that the set $z_1 \dots z_4$ is linearly dependent.

Next, we show that, under certain circumstances, we can confine our attention to the case $n \leqslant 4$. Let $z_1, \ldots z_n$ and $\zeta_1, \ldots \zeta_n$ be two sets of vectors such that any $r+1$ element subset of either is linearly dependent, and the matrices $z_j \cdot z_k$ and $\zeta_j \cdot \zeta_k$, $j, k = 1, 2, \ldots n$, are equal and of rank r. For convenience, we may suppose that $G(z_1, \ldots z_r) \neq 0$. Expand the z_j, $j = r+1, \ldots n$ in terms of the z_j, $j = 1, \ldots r$.

$$z_j = \sum_{l=1}^{r} \alpha_{jl} z_l, \quad j = r+1, \ldots n. \tag{33}$$

The α_{jl} are expressible in terms of scalar products since they are the solutions of the linear equations

$$z_k \cdot z_j = \sum_{l=1}^{r} \alpha_{jl} z_k \cdot z_l, \quad \begin{array}{l} j = r+1, \ldots n \\ k = 1, \ldots r. \end{array} \tag{34}$$

The equations (34) have a unique solution because $G(z_1, \ldots z_r) \neq 0$. Thus, we see that, if a Lorentz transformation Λ can be found satisfying (29) or (30) for $j = 1, \ldots r$, it will also satisfy them for $j = r+1, \ldots n$, provided that the rank of $z_i \cdot z_j = \zeta_i \cdot \zeta_j$ is r and there are at most r linearly independent z_i and ζ_i. These last provisos are *always* satisfied if $r = 4$ or 3. For $r = 4$, we have just established the equivalence of $G(z_1 \ldots z_4) \neq 0$ and linear independence of the vectors z_1, z_2, z_3, z_4. By the very same argument, it cannot happen that the rank of $z_i \cdot z_j$ is three and the number of linearly independent z_j is four. The fact that there can be "extra" linearly independent vectors when the rank of $z_i \cdot z_j$ is one or two is the source of the possibility that (30), but not (29), may hold.

Now we will construct a Λ satisfying (29) under the assumption of the preceding paragraph, i. e., that there is at most a linearly independent set of r z's and at most a linearly independent set of r ζ's and the matrix $z_i \cdot z_j = \zeta_i \cdot \zeta_j$ has rank r. Our preceding considerations assure us that it suffices to consider the case $n \leqslant 4$. The z_i, $i = 1, \ldots n$ span an r-dimensional linear manifold, M. For convenience, we let the $z_1, \ldots z_r$ be a linearly independent set. Let the corresponding r-dimensional manifold spanned by the ζ_i, $i = 1, \ldots n$ be devoted by N. The orthogonal manifolds $M^\perp$ and $N^\perp$ respectively, are $4-r$ dimensional and the intersections $M \cap M^\perp$ and $N \cap N^\perp$ contain only the vector zero. A proof of these last statements is obtained as follows. Supplement the vectors $z_1, \ldots z_r$ by $z'_1, \ldots z'_{4-r}$ and the vectors $\zeta_1, \ldots \zeta_r$ by $\zeta'_1, \ldots \zeta'_{4-r}$ so that the resulting sets are bases for the whole four-dimensional space. Then the r equations

$$\sum_{l=1}^{r} \alpha_l (z_l \cdot z_j) + \sum_{l=1}^{4-r} \alpha_{r+l} (z_l' \cdot z_j) = 0 \quad j = 1, \ldots r,$$

have rank r, so they have $4-r$ linearly independent solutions

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Because the basis vectors are linearly independent, the vectors

$$\sum_{l=1}^{r} \alpha_l z_l + \sum_{l=1}^{4-r} \alpha_{l+r} z_l'$$

constitute a linearly independent $4-r$ element set. Consequently, $M^\perp$ has dimension $4-r$. That $M \cap M^\perp$ contains no non-zero vector is equivalent to the statement that the equations

$$\sum_{l=1}^{r} \alpha_l (z_l \cdot z_j) = 0 \quad j = 1, \ldots r \tag{35}$$

have no non-trivial solution. The analogous statements for $N^\perp$ and $N \cap N^\perp$ are obtained by replacing z_i by ζ_i in the above proof.

Notice that the Gram determinant of the entire basis

$$z_1, \ldots z_r, z_1', \ldots z_{4-r}'$$

is the product of the Gram determinants of the sets $z_1, \ldots z_r$ and $z_1', \ldots z_{4-r}'$ so that the Gram determinant of $z_1', \ldots z_{4-r}'$ is non-zero. A similar statement holds for the $\zeta_1', \ldots \zeta_{4-r}'$, and we want to use these facts to show that new bases $z_1'', \ldots z_{4-r}''$ and $\zeta''_1, \ldots \zeta_{4-r}''$ for $M^\perp$ and $N^\perp$, respectively, can be chosen so that

$$z_i'' \cdot z_j'' = \delta_{ij} = \zeta_i'' \cdot \zeta_j'' \quad i, j = 1, \ldots 4-r. \tag{36}$$

Consider $M^\perp$. Since the Gram determinant of the $z_1', \ldots z_{4-r}'$ does not vanish, some scalar product of these vectors does not vanish, and, consequently, there is at least one vector of non-zero length in $M^\perp$. Adjust its length to 1 and call it z_1''. By induction, using the arguments of this and the immediately preceding paragraph, we can construct $z_1'', z_2'', \ldots z_{4-r}''$ orthogonal to each other and of length one. An analogous construction holds for the $\zeta_1'', \zeta_2'', \ldots \zeta_{4-r}''$. It is evidently crucial for the success of the construction that at each stage the relevant Gram determinants are non-zero[7].

Now we are in a position to define the Λ required by (29) as the complex linear transformation determined by the equations

$$\Lambda z_j = \zeta_j \qquad j = 1, \ldots r,$$
$$\Lambda z_j'' = \zeta_j'' \qquad j = 1, \ldots 4-r.$$

Λ so defined preserves scalar products by virtue of (35) and (36):

$$\left.\begin{aligned}
&\left[\Lambda\left(\sum_{j=1}^{r}\alpha_j z_j + \sum_{j=1}^{4-r}\alpha_{j+r} z_j''\right)\right]\cdot\left[\Lambda\left(\sum_{j=1}^{r}\beta_j z_j + \sum_{j=1}^{4-r}\beta_{j+r} z_j''\right)\right]\\
&= \sum_{j,k=1}^{r}\alpha_j\beta_k(\Lambda z_j)\cdot(\Lambda z_k) + \sum_{j,k=1}^{4-r}\alpha_{j+r}\beta_{k+r}(\Lambda z_j'')\cdot(\Lambda z_k'')\\
&= \sum_{j,k=1}^{r}\alpha_j\beta_k\zeta_j\cdot\zeta_k + \sum_{j,k=1}^{4-r}\alpha_{j+r}\beta_{k+r}\zeta_j''\cdot\zeta_k''\\
&= \sum_{j,k=1}^{r}\alpha_j\beta_k z_j\cdot z_k + \sum_{j,k=1}^{4-r}\alpha_{j+r}\beta_{j+r} z_j''\cdot z_k''\\
&= \left(\sum_{j=1}^{r}\alpha_j z_j + \sum_{j=1}^{4-r}\alpha_{j+r} z_j''\right)\cdot\left(\sum_{j=1}^{r}\beta_j z_j + \sum_{r=1}^{4-r}\beta_{j+r} z_j''\right).
\end{aligned}\right\} \tag{37}$$

It is therefore the required Lorentz transformation, and the proof of Lemma 2 is complete for the rank three and four cases.

For rank two and one we must deal with the cases in which the number of linearly independent z_j (or ζ_j) is larger than the rank of the matrix $z_j\cdot z_k$. That we cannot expect to find a Λ satisfying (29) in this case is clear from the example $z_1 = (1, 0, 0, 0)$, $z_2 = (1, 1, i, 0)$; $\zeta_1 = (1, 0, 0, 0)$, $\zeta_2 = (1, 0, 0, 0)$ of vectors satisfying (28) with a matrix of rank one; a Λ certainly cannot carry linearly independent vectors (the z's) into linearly dependent vectors (the ζ's).

Let Z be of rank $r = 1$, or 2, and let $n > r$. There is a subset of r vectors z_i with non-vanishing Gram determinant. They span a subspace which we call M_1. It is a subspace of M, the subspace spanned by all z_i', $i = 1, \ldots n$, whose dimension we denote by m. The corresponding subspaces for the vectors ζ_i, we denote by N_1 (of dimension r) and N (of dimension m'). Because M_1 and N_1 have non-vanishing Gram determinants there is a unique decomposition of the vectors z_i and ζ_i:

$$z_i = z_i' + z_i'' \quad i = 1, \ldots n,\; z_i' \,\varepsilon\, M_1,\; z_i'' \,\varepsilon\, M_1^{\perp}\cap M$$
$$\zeta_i = \zeta_i' + \zeta_i'' \quad i = 1, \ldots n,\; \zeta_i' \,\varepsilon\, N_1,\; \zeta_i'' \,\varepsilon\, N_1^{\perp}\cap N.$$

Now the rank of the matrices $z_i''\cdot z_j''$, $\zeta_i''\cdot\zeta_j''$, $i, j = 1, \ldots n$ must be zero, since the rank of $z_i'\cdot z_j'$ and $\zeta_i'\cdot\zeta_j'$ is already r. Furthermore, the subspaces $M_1^\perp \cap M$ and $N_1^\perp \cap N$ of the $4-r$ dimensional spaces $M_1^\perp$, and $N_1^\perp$ can at most be one-dimensional since they are isotropic and $M_1^\perp$ and $N_1^\perp$ have non-vanishing Grammian[7]. Thus, $z_i'' = \gamma_i \omega'$, $i = 1, \ldots n$, $\gamma_1, \ldots \gamma_n$ complex numbers and $(\omega')^2 = 0$, $\omega' \cdot z_i' = 0$, $i = 1, \ldots n$, and similarly $\zeta_i'' = \delta_i \omega$, $\omega^2 = 0 = \omega \cdot \zeta_i'$, $i = 1, \ldots n$. Incidentally, we see that the dimensions m and m' of M and N are $r+1$ at most.

Now we choose an orthonormal basis ω_i, $i = 1, \ldots 4-r$ for $M_1^\perp$ and η_i, $i = 1, \ldots 4-r$ for $N_1^\perp$ so that ω' and ω lie in the subspaces spanned by ω_1, ω_2 and η_1, η_2 respectively. For $r = 2$, no construction is required to obtain this property of ω's and η's. For $r = 1$, we know the construction is possible because there has to be at least one vector of non-zero length orthogonal to ω' (or ω) and we can take it suitably normalized to be ω_3 (or η_3). Since ω' is a linear combination of ω_1 and ω_2 and of zero length, it must be of the form $a(\omega_1 \pm i\omega_2)$ and by changing the sign of ω_2, if necessary, we can arrange it so that $\omega' = a(\omega_1 + i\omega_2)$. Similarly, $\omega = b(\eta_1 + i\eta_2)$.

Finally, the Lorentz transformation Λ required by (30) is the linear transformation defined by

$$\begin{aligned} \Lambda z_i' &= \zeta_i' \quad \text{for the } r \text{ vectors } z_i' \text{ and } \zeta_i' \text{ which span } M_1 \text{ and } N_1 \\ \Lambda \omega_i &= \eta_i \quad i = 1, 2, \ldots, 4-r. \end{aligned}$$

That this is indeed a Lorentz transformation follows by an argument like that used for the higher ranks. That Λ satisfies (30) follows from the computation

$$\begin{aligned} \Lambda z_k &= \Lambda z_k' + \gamma_k a(\Lambda\omega_1 + i\Lambda\omega_2) \\ &= (\zeta_k - \delta_k \omega) + \gamma_k a(\eta_1 + i\eta_2) \\ &= \zeta_k + (\gamma_k a - \delta_k b)(\eta_1 + i\eta_2) \qquad k = 1, \ldots n. \end{aligned}$$

Therefore, when the rank r of $z_i \cdot z_j$ is one or two, there exists a Lorentz transformation, Λ, such that the Λz_i and ζ_i differ by multiples of a fixed vector orthogonal to all Λz_i and ζ_i, i $= 1, \ldots n$, and of length zero. This completes the proof.

To round out the information provided by Lemma 2, we make three additional remarks. First of all, if Z is an arbitrary complex symmetric $n \times n$ matrix of rank $r \leqslant 4$, it can be written in the form

$$Z_{ij} = \zeta_i \cdot \zeta_j, \; i, j = 1, \ldots n, \tag{38}$$

where $\zeta_1, \ldots \zeta_n$ are four vectors which span a linear manifold M of dimension r. This follows immediately from the standard theorem of algebra which says that, if Z is a complex symmetric $n \times n$ matrix of rank r, there exists a non-singular $n \times n$ matrix S such that

$$Z = S \left\{ \begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right\} S^{\mathsf{T}}, \tag{39}$$

where the one in the upper left hand corner of the matrix indicated in curly brackets stands for the $r \times r$ unit matrix[8]. One can then take the components $\zeta_j{}^\mu$ as $S_{j1}, iS_{j2}, \ldots iS_{jr}$ followed by $4-r$ zeros. Equation (39) then reduces to (38). At most r of these vectors can be linearly independent and, in fact, exactly r are because otherwise Z would have rank less than r.

Our second remark is that, if the point $\zeta_1 + \alpha_1\omega, \ldots \zeta_n + \alpha_n\omega$ is in the tube and $\omega^2 = 0 = \omega \cdot \zeta_j$, $j = 1, \ldots n$, then ω and ζ_j are of the form

$$\omega = a(\omega_1 + i\omega_2), \text{ where } \omega_1 \text{ and } \omega_2 \text{ are real and}$$
$$\omega_1^2 = \omega_2^2 = -1, \; \omega_1 \cdot \omega_2 = 0,$$
$$\zeta_j = \zeta_j' + \beta_j\,\omega, \text{ where } \zeta_j' \cdot \omega_1 = 0 = \zeta_j' \cdot \omega_2, j = 1, \ldots n,$$
$$\text{and } \zeta_1' \ldots \zeta_n' \text{ is a point of the tube.}$$

To prove these statements, we split ω into its real and imaginary parts: $\omega = q + ir$. Then, $\omega^2 = 0$ implies $q^2 = r^2$ and $q \cdot r = 0$ so that q and r are either light-like and collinear or space-like and orthogonal. The first alternative cannot occur, because the requirement $\omega \cdot \zeta_j = 0$ would then force q and r to be orthogonal to a time-like vector, and the first half of (40) follows if we choose ω_1 and ω_2 as q and r normalized to length minus one. The second half is easily seen if the real and imaginary parts of ζ_j are expanded in terms of ω_1 and ω_2 as follows:

$$\left.\begin{array}{l} \xi_j = \varrho_1^{(j)}\omega_1 + \varrho_2^{(j)}\omega_2 + \xi_j', \quad \text{where } \xi_j' \cdot \omega_1 = 0 = \xi_j' \cdot \omega_2, \; j = 1, \ldots n, \\ \eta_j = \sigma_1^{(j)}\omega_1 + \sigma_2^{(j)}\omega_2 + \eta_j', \text{ where } \eta_j' \cdot \omega_1 = 0 = \eta_j' \cdot \omega_2, \; j = 1, \ldots n. \end{array}\right\} \tag{40}$$

The orthogonality condition $(\xi_j - i\eta_j)\cdot\omega = 0$ then leads directly to $\sigma_1^{(j)} = \varrho_2^{(j)}$ and $\sigma_2^{(j)} = -\varrho_1^{(j)}$, and therefore

$$\zeta_j = \xi_j - i\eta_j = \xi_j' - i\eta_j' + (\varrho_1^{(j)} - i\sigma_1^{(j)})(\omega_1 + i\omega_2).$$

If we write $\zeta_j' = \xi_j' - i\eta_j'$, it remains to verify that the point $\zeta_1', \ldots \zeta_n'$ is in the tube. Because $\zeta_1 + \alpha_1, \omega, \ldots \zeta_n + \alpha_n\omega$ lies in the tube by assumption, the squares of the lengths of the imaginary parts of $\zeta_j + \alpha_j\omega$, $j = 1, \ldots n$ are positive:

$$[\eta_j' - (\beta_j''\omega_1 + \beta_j'\omega_2)]^2 = \eta'^2 - (\beta_j'')^2 - (\beta_j')^2 > 0, \tag{41}$$

where $(\beta_j + \alpha_j)\; a = \beta_j' + i\beta_j''$ with β_j' and β_j'' real, and β_j and a defined in (40). Clearly, (41) implies $\eta_j'^2 > 0$. Furthermore, β_j'' and β_j' can be continuously decreased to zero without (41) losing its validity. Consequently, if the imaginary part of $\zeta_j + \alpha_j \omega$ points into the forward cone, so must that of ζ'_j, so that the point $\zeta'_1, \ldots \zeta'_n$ is in the tube.

Our third remark is that, if ζ'_j, $j = 1, \ldots n$ and ω have the properties $\omega = a(\omega_1 + i\omega_2)$, with ω_1 and ω_2 real and ${\omega^2}_1 = -1 = {\omega_2}^2$, $\omega_1 \cdot \omega_2 = 0$, $\zeta'_j \cdot \omega_1 = 0 = \zeta'_j \cdot \omega_2$ for $j = 1, \ldots n$, then there exists a one-parameter family of Lorentz transformations $\Lambda(\theta)$, $0 \leqslant \theta \leqslant \infty$ with the properties $\Lambda(\theta)\zeta'_j = \zeta'_j$, $j = 1, \ldots n$ and $\Lambda(\theta)\omega = e^{-\theta}\omega$. The transformation $\Lambda(\theta)$ is defined as the identity on vectors orthogonal to ω_1 and ω_2, but

$$\left.\begin{aligned} \Lambda(\theta)\omega_1 &= \omega_1 \cos i\theta - \omega_2 \sin i\theta \\ \Lambda(\theta)\omega_2 &= \omega_1 \sin i\theta + \omega_2 \cos i\theta . \end{aligned}\right\} \tag{42}$$

With its definition completed by linearity, $\Lambda(\theta)$ is a Lorentz transformation with the required properties.

Two important consequences follow immediately from these remarks. The first is that, if a point $\zeta_1 + \alpha_1\omega, \ldots \zeta_n + \alpha_n\omega$ with $\omega^2 = 0$, $\omega \cdot \zeta_j = 0$, $j = 1, \ldots n$ lies in the extended tube, then all points of the form $\zeta_1 + \alpha'_1\omega$, $\ldots \zeta_n + \alpha'_n\omega$ do also, where $\alpha'_1, \ldots \alpha'_n$ are arbitrary complex numbers. Since $\zeta_1 + \alpha_1\omega, \ldots \zeta_n + \alpha_n\omega$ is in the extended tube, there exists a complex Lorentz transformation Λ such that $\Lambda\zeta_1 + \alpha_1\Lambda\omega, \ldots \Lambda\zeta_n + \alpha_n\Lambda\omega$ is in the tube. Using the second remark, we may then write $\Lambda\zeta_j + \alpha'_j\Lambda\omega$ as $\Lambda\zeta'_j + (\beta_j + \alpha'_j)\Lambda\omega$, where $\Lambda\zeta'_1, \ldots \Lambda\zeta'_n$ is a point of the tube, and $\Lambda\omega = a(\omega_1 + i\omega_2)$, $\omega_1^2 = -1 = \omega_2^2$ and $\omega_1 \cdot \omega_2 = \Lambda\zeta'_j \cdot \omega_1 = \Lambda\zeta'_j \cdot \omega_2 = 0$, $j = 1, \ldots n$. Using the third remark, we obtain a family of transformations $\Lambda(\theta)$ such that

$$\Lambda(\theta)[\Lambda\zeta_j + \alpha'_j\Lambda\omega] = \Lambda\zeta'_j + e^{-\theta}(\beta_j + a'_j)\Lambda\omega,\ j = 1, \ldots n.$$

These equations say that the point $\Lambda\zeta_1 + \alpha'_1\Lambda\omega, \ldots \Lambda\zeta_n + \alpha'_n\Lambda\omega$ can be brought arbitrarily near to the point $\Lambda\zeta'_1, \ldots \Lambda\zeta'_n$ (which lies in the tube) by a complex Lorentz transformation. Therefore, the point $\Lambda\zeta_1 + \alpha'_1\Lambda\omega$, $\ldots \Lambda\zeta_1 + \alpha'_n\Lambda\omega$ and, consequently, the point $\zeta_1 + \alpha'_1\omega, \ldots \zeta_n + \alpha'_n\omega$ lie in the extended tube.

The second consequence is that an invariant analytic function (satisfying the hypotheses of Lemma 1) is necessarily single-valued on $\mathfrak{M}_n$. For points of $\mathfrak{M}_n$ where the rank of $z_i \cdot z_j$ $i, j = 1, \ldots n$ is three or min $(3, n)$ the statement follows immediately, because $f(z_1, \ldots z_n) = f(\zeta_1, \ldots \zeta_n)$ is a consequence of $z_i \cdot z_j = \zeta_i \cdot \zeta_j$, $i, j = 1, \ldots n$, since, by Lemma 2 there exists a $\Lambda \varepsilon \mathfrak{L}$ such that $\Lambda z_j = \zeta_j$, $j = 1, \ldots n$. For $z_i \cdot z_j$, $i, j = 1, \ldots n$, of rank 2, $n > 2$, or 1, $n > 1$, we know by Lemma 2 that $z_i \cdot z_j = \zeta_i \cdot \zeta_j$ implies $\Lambda z_j = \zeta_j + \alpha_j\omega$, $j = 1, \ldots n$, so to show the single valuedness of f at such points of $\mathfrak{M}_n$ it suffices to show that

$$f(\zeta_1, \ldots \zeta_n) = f(\zeta_1 + \alpha_1 \omega, \ldots \zeta_n + \alpha_n \omega). \tag{43}$$

Furthermore, without loss of generality, we may restrict ourselves to the case in which ζ_j and ω have the properties $\omega = a(\omega_1 + i\omega_2)$ with ω_1 and ω_2 real and $\omega_1^2 = \omega_2^2 = -1$, $\omega_1 \cdot \omega_2 = 0$, and $\zeta_j \cdot \omega_1 = 0 = \zeta_j \cdot \omega_2$, $j = 1, \ldots n$. By introducing the $\Lambda(\theta)$ defined in our third remark above, we get

$$\begin{aligned} & f(\zeta_1, \ldots \zeta_n) - f(\zeta_1 + \alpha_1 \omega, \ldots \zeta_n + \alpha_n \omega) \\ &= f(\Lambda(\theta)\zeta_1, \ldots \Lambda(\theta)\zeta_n) - f(\Lambda(\theta)\zeta_1 + \alpha_1 \Lambda(\theta)\omega, \ldots \Lambda(\theta)\zeta_n + \alpha_n \Lambda(\theta)\omega) \\ &= f(\zeta_1, \ldots \zeta_n) - f(\zeta_1 + \alpha_1 e^{-\theta}\omega, \ldots \zeta_n + \alpha_n e^{-\theta}\omega). \end{aligned}$$

From the continuity of f at $\zeta_1, \ldots \zeta_n$, we see that the last expression vanishes in the limit as $\theta \to \infty$, which proves (43).

Now we turn to the connection between the topology of the vectors $z_1, \ldots z_n$ and the topology on $\mathfrak{M}_n$.

Lemma 3.

Let Z be an $n \times n$ complex symmetric matrix of rank r, $1 \leqslant r \leqslant 4$ and ω an arbitrary real positive number. Then there exists a set of n four vectors $z_1, \ldots z_n$ and a neighbourhood of them consisting of the four-vectors $z_1 + v_1$, $\ldots z_n + v_n$ with

$$|v_j^\mu| < \omega \quad j = 1, \ldots n; \; \mu = 0, 1, 2, 3 \tag{44}$$

such that $Z_{ij} = z_i \cdot z_j$, $i, j = 1, \ldots n$ and the matrices, Z'_{ij}, defined by

$$Z'_{ij} = (z_i + v_i) \cdot (z_j + v_j), \; i. j, = 1 \ldots n, \tag{45}$$

cover a neighbourhood of Z in the set of complex symmetric matrices of rank $\leqslant 4$, i.e., for suitably chosen $\eta > 0$, every complex symmetric matrix Z' of rank $\leqslant 4$ which satisfies

$$|Z_{ij} - Z'_{ij}| < \eta \qquad i, j = 1, \ldots n$$

is of the form (45) with v_j satisfying (44).

Proof.

The direct determination of the vectors v_i, $i = 1, \ldots n$ satisfying (45) would be somewhat involved, so we make a series of transformations to reduce the problem to a simpler one.

We know from the first remark following Lemma 2 that the matrix Z may be written as a matrix of scalar products: $Z_{ij} = z_i \cdot z_j$, $i, j = 1, \ldots n$, where, if Z has rank r, the vectors z_j, $j = 1, \ldots n$ span a linear manifold of dimension r. That being the case, there exists an r-element linearly independent subset of the n vectors which may as well, for convenience, be taken as $z_1, \ldots z_r$. A new set of z's, which we denote by z'_j, $j = 1, \ldots n$, is defined by

$$z'_j = z_j \quad j = 1, \ldots r$$

$$0 = z'_j = z_j - \sum_{k=1}^{r} \alpha_{jk} z_k, \quad j = r+1, \ldots n.$$

This linear transformation from z's to z''s has determinant 1 and is therefore non-singular. Subsequent to this transformation, we carry out a linear transformation on the subspace M which normalizes and orthogonalizes the z'_j, $j = 1, \ldots r$. The product of these two transformations is given by a matrix A which has the property

$$AZA^{\mathsf{T}} = \begin{array}{c} \\ r \end{array}\!\!\begin{array}{c} r \\ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}.$$

Now, since A is non-singular, it maps neighbourhoods of $z_1, \ldots z_n$ into neighbourhoods of $\sum_{j=1}^{n} A_{1j} z_j, \ldots \sum_{j=1}^{n} A_{nj} z_j$ and neighbourhoods of Z into neighbourhoods of AZA^{T} in an invertible manner. Thus, it suffices to prove Lemma 3, in the case that the first r of the vectors z_j are orthonormal and the rest are zero.

This first simplification of the problem uses a transformation, A, which depends only on the z_j, but not on which point in the neighbourhood of the z_j is under consideration. The second transformation we make will be different for each Z', and makes the first r of the vectors $z_j + v_j$ orthogonal to the rest.

Define a new set of v_j by the equations

$$v'_j = v_j \quad j = 1, \ldots r,$$

$$v'_j = v_j - \sum_{k=1}^{r} \beta_{jk} (z_k + v_k) \quad j = r+1, \ldots n,$$

where the numbers β_{jk}, $j = r+1, \ldots n$, $k = 1, \ldots r$, are determined from the condition

$$v'_j \cdot (z_k + v_k) = 0, \; j = r+1, \ldots n, \; k = 1, \ldots r. \tag{46}$$

An elementary calculation yields

$$\beta_{jl} = \sum_{k=1}^{r} Z'_{jk} (Z'^{-1})_{kl} \qquad j = r+1, \ldots n, \; l = 1, \ldots r,$$

where the indicated matrix inverse means the inverse of the $r \times r$ matrix Z'_{jk}, $j, k = 1, \ldots r$. The simplification of the problem achieved in the preceding paragraph enables us to write $Z' = Z + B$ where $Z_{jk} = \delta_{jk}, j, k = 1, \ldots r$, $Z_{jk} = 0$, $j, k > r$. When the matrix elements of B satisfy $|B_{jk}| < \eta, j, k = 1, \ldots n$ and η is sufficiently small, the estimates $|Z'_{jk}| < \eta; j = r+1, \ldots n, k = 1, \ldots r$, and $|(Z'^{-1})_{kl} - \delta_{kl}| < \eta\,(1 - r\eta)^{-1}$, $k, l = 1, \ldots r$, hold, so that the transformation from the v's to the v''s has an inverse and carries small neighbourhoods of the origin in v space into small neighborhoods in v' space, and vice versa. Thus the problem has been reduced to that of finding $v'_1, \ldots v'_n$ satisfying

$$\left(\begin{array}{c|c} (z_j + v'_j)\cdot(z_k + v'_k) & (z_j + v'_j)\cdot v'_k \\ \hline v'_j\cdot(z_k + v'_k) & v'_j \cdot v'_k \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right), \tag{47}$$

where the dividing line in the matrix is at the r^{th} row and column and

$$(B_1)_{jk} = B_{jk}, \quad j, k = 1, \ldots r, \text{ while}$$

$$(B_2)_{jk} = B_{jk} - \sum_{l,\, m=1}^{r} (Z')_{jl} (Z'^{-1})_{lm} (Z')_{mk} \quad j, k = r+1, \ldots n.$$

When $|B_{jk}| < \eta$, $j, k = r+1, \ldots n$ we have $|(B_2)_{jk}| \leqslant \eta\,(1 - r\eta)^{-1}$.

The simple expedient of requiring v'_k, $k = 1, \ldots r$ to have vanishing components beyond the r^{th} and v'_k, $k = r+1, \ldots n$, vanishing components before the $(r+1)$st in a basis in which the vectors z_k, $k = 1, 2, \ldots r$, are the first r coordinate axes guarantees the orthogonality relations (46). The problem of satisfying the upper left hand corner of the relation (47) then reads in $r \times r$ matrix form

$$(1 + v')\,(1 + v')^{\mathsf{T}} = 1 + B_1,$$

where $1 + v'$ is the $r \times r$ matrix whose k^{th} row is composed of the components of $z_k + v'_k$. By making the special choice of the components of v'_k, $k = 1, 2 \ldots r$, which makes $1 + v'$ symmetric, we are led to the solution

$$1 + v' = [1 + B_1]^{\frac{1}{2}},$$

the right hand side being defined by its power series about the matrix $B_1 = 0$. This series converges for sufficiently small η and leads to components of v' which satisfy $|(v')_j^{\mu}| \leqslant \eta\,(1 - r\eta)^{-1}$, $j = 1, \ldots r$.

To be sure that the lower right hand corner problem has a solution in terms of vectors with only $4 - r$ components, we have to be sure that the rank of B_2 is $\leqslant 4 - r$. This follows immediately by an argument which we used several times in the proof of Lemma 2. The right hand side of (47) is a matrix of rank $\leqslant 4$. The $r \times r$ matrix $1 + B_1$ is of rank r, therefore B_2 is of rank $\leqslant 4 - r$. (Compute the determinant of all principal minors of the right hand side of (47), which have $1 + B_1$ in their upper left hand corner. They vanish if they have more than four rows and columns.) Then, by the theorem quoted in (39), we know v'_k, $k = r+1, \ldots n$, exist such that $v'_k \cdot v'_l = (B_2)_{kl}$, $k, l = r+1, \ldots n$. Of course, there is a variety of sets of vectors v'_k, $k = r+1, \ldots n$ satisfying this last relation. We have to be sure that sets can be chosen so that their components are uniformly small when the matrix elements of B_2 are small. It can be shown that the v'_k can always be chosen so that*

$$|v'^{\mu}_k| < 8^{\frac{1}{2}[(4-r)-1]} [\sup_{i,j} |(B_2)_{ij}|]^{\frac{1}{2}}. \tag{48}$$

Collecting the estimates of the v'_k and $|(B_2)_{ij}|$ we see that, when η is sufficiently small, components of the v_i satisfying (45) can always be chosen so as to satisfy (44).

Lemmas 1 and 2 enabled us to prove that an invariant analytic function, f, (satisfying the hypotheses of the theorem) is necessarily a single-valued function on $\mathfrak{M}_n$. The continuity of f on $\mathfrak{M}_n$ is an immediate consequence of Lemma 3, because it demonstrates that small neighbourhoods of any point, P, on $\mathfrak{M}_n$ can have pre-images in the space of vectors which are small neighbourhoods of a pre-image of P.

We now turn to the proof of the analyticity of f on $\mathfrak{M}_n$.

Lemma 4.

If $f(z_1, \ldots z_n)$ is a function of the vector variables $z_1, \ldots z_n$ analytic in the tube (extended tube) and invariant under transformations of $L^{\uparrow}$, then the following equations are satisfied at every point of the tube (extended tube).

$$\sum_{j=1}^{n} \left(z_{j\mu} \frac{\partial f}{\partial z_j{}^{\nu}} - z_{j\nu} \frac{\partial f}{\partial z_j{}^{\mu}} \right) = 0. \tag{49}$$

Proof.

Let $\Lambda(a)$, $-\infty < a < \infty$, be any one parameter subgroup of real Lorentz transformations, and $z_1, \ldots z_n$ a point of the tube (extended tube). Differentiating the identity

* The inequality (48) can be proved by going through the classical induction proof of (39) estimating the size of each term. We are indebted to V. Bargmann for pointing out (48), as well as showing us a version of the proof of this Lemma which we have followed rather closely.

$$f(\Lambda(a)z_1, \ldots \Lambda(a)z_n) = f(z_1, \ldots z_n)$$

with respect to a, we obtain

$$\frac{df}{da} = \sum_{j=1}^{n} \frac{\partial f}{\partial(\Lambda(a)z_j)^\mu} \frac{\partial(\Lambda(a)z_j)^\mu}{\partial a} = 0.$$

At $a = 0$, we have

$$\left.\frac{\partial(\Lambda(a)z_j)^\mu}{\partial a}\right|_{a=0} = \lambda^\mu{}_\nu\, z_j{}^\nu,$$

where $\lambda^\mu{}_\nu$ is a real 4×4 matrix satisfying

$$\lambda^\mu{}_\nu = -\lambda_\nu{}^\mu, \tag{50}$$

and defining the one parameter subgroup. Hence

$$\sum_{j=1}^{n} \lambda^\mu{}_\nu\, z_j{}^\nu \frac{\partial f}{\partial z_j{}^\mu} = 0. \tag{51}$$

Now, any real 4×4 matrix satisfying (50) generates a one parameter subgroup, so we may take λ to have zero matrix elements except for a fixed pair $\lambda_{\mu\nu}$ and $\lambda_{\nu\mu}$, then (51) reduces to (49) and the Lemma is proved.

Considered for fixed z_k, the equations (49) are a set of linear equations in the $4\,n$ unknowns $\frac{\partial f}{\partial z_j{}^\mu}$, $j = 1, \ldots n$, $\mu = 0, 1, 2, 3$. There are at most 6 independent equations. The derivatives of any invariant function must satisfy this set of equations at each point of the tube (extended tube). Of course, the coefficient matrix of the equations varies from point to point.

Lemma 5.

If at a point, $z_1, \ldots z_n$, of the extended tube, the number of linearly independent solutions of the equations (49) is ϱ and this ϱ dimensional linear manifold of solutions is spanned by the solutions which come from ϱ scalar products, then any invariant analytic function, f, may be represented in a neighbourhood of $z_1, \ldots z_n$ of the extended tube as a convergent power series in the ϱ scalar products, i.e., f is an analytic function of the ϱ scalar products at $z_1, \ldots z_n$.

Proof.

In the customary nomenclature, a set of analytic functions $f^{(i)}$ $i = 1, \ldots m \leqslant 4\,n$ of the four-vector variables $z_1, \ldots z_n$ is functionally independent at the point $z_1, \ldots z_n$ if the Jacobian matrix $\frac{\partial f^{(i)}}{\partial z_j^{\mu}}$ ($i = 1, \ldots m$ labels rows; $j = 1, \ldots n$, $\mu = 0, 1, 2, 3$ label columns) has rank m at $z_1, \ldots z_n$. In other words, the m rows of the Jacobian matrix regarded as $4\,n$ component vectors with components labeled by j and μ are linearly independent. Since functional independence at a point is equivalent to the requirement that the determinant of some $m \times m$ minor of the Jacobian matrix be different from zero, functional independence at a point implies functional independence in some neighbourhood of the point.

In this terminology, the hypothesis of the Lemma is that ϱ functionally independent scalar products exist. We shall denote these scalar products by $f^{(i)}$, $i = 1, \ldots \varrho$. Since the Jacobian matrix has rank ϱ, there exists a ϱ element subset, T, of the $4\,n$ variables z_j^{μ}, $j = 1, \ldots n$, $\mu = 0, 1, 2, 3$, such that the determinant of the square matrix

$$\frac{\partial f^{(i)}}{\partial z_j^{\mu}} \quad i = 1, \ldots \varrho, \; z_j^{\mu} \,\varepsilon\, T$$

is non-zero.

Then, by the implicit function theorem for several complex variables,* the $z_j^{\mu} \,\varepsilon\, T$ are analytic functions of $f^{(i)}$ $i = 1, \ldots \varrho$ in a neighbourhood of $f^{(i)}(z_1, \ldots z_n)$ and

$$f(z_1, \ldots z_n) = g\,(f^{(1)}, \ldots f^{(\varrho)}, h^{(1)}, \ldots h^{(4n-\varrho)}),$$

where g is analytic in the variables $f^{(i)}$, $i = 1, \ldots \varrho$ and $h^{(j)}$, $j = 1, \ldots 4\,n - \varrho$ in a neighbourhood of

$$f^{(1)}(z_1, \ldots z_n), \ldots f^{(\varrho)}(z_1, \ldots z_n), h^{(1)}(z_1, \ldots z_n), \ldots h^{(4n-\varrho)}(z_1, \ldots z_n).$$

The variables $h^{(j)}$ are the z_k^{μ} which are not in T. The variables

$$f^{(i)}, \; i = 1, \ldots \varrho \text{ and } h^{(j)}, \; j = 1, \ldots 4\,n - \varrho$$

are obviously functionally independent in a neighbourhood of $z_1, \ldots z_n$.

The derivatives of f can now be expressed in terms of the derivatives of g as follows:

$$\frac{\partial f}{\partial z_j^{\mu}} = \sum_{i=1}^{\varrho} \frac{\partial g}{\partial f^{(i)}} \frac{\partial f^{(i)}}{\partial z_j^{\mu}} + \sum_{k=1}^{4n-\varrho} \frac{\partial g}{\partial h^{(k)}} \frac{\partial h^{(k)}}{\partial z_j^{\mu}}.$$

However, according to the hypothesis of the Lemma, at the point $z_1, \ldots z_n$

* See B and M, p. 39, theorem 9.

$$\frac{\partial f}{\partial z_j^\mu} = \sum_{i=1}^{\varrho} \alpha_i \frac{\partial f^{(i)}}{\partial z_j^\mu}.$$

Hence,

$$\sum_{i=1}^{\varrho} \left(\frac{\partial g}{\partial f^{(i)}} - \alpha_i \right) \frac{\partial f^{(i)}}{\partial z_j^\mu} + \sum_{l=1}^{4n-\varrho} \frac{\partial g}{\partial h^{(i)}} \frac{\partial h^{(i)}}{\partial z_j^\mu} = 0.$$

But regarded as vectors with $4n$ components, $\frac{\partial f^{(i)}}{\partial z_j^\mu}$, $i = 1, \ldots \varrho$ and $\frac{\partial h^{(i)}}{\partial z_j^\mu}$, $i = 1, \ldots 4n - \varrho$ are linearly independent. (That is what the functional independence of the $f^{(i)}$ and $h^{(i)}$ means.) Therefore, their coefficients in this equation must vanish. In particular,

$$\frac{\partial g}{\partial h^{(i)}} = 0, \quad i = 1, \ldots 4n - \varrho,$$

so g is independent of the $h^{(i)}$ and the Lemma is proved.

Lemma 6.

Let N be the maximum number of linearly independent vectors contained in the set of 4-vectors $z_1, \ldots z_n$, then the set of six linear equations

$$\sum_{j=1}^{n} (z_{j\mu} X_{j\nu} - z_{j\nu} X_{j\mu}) = 0 \quad \mu, \nu = 0, 1, 2, \quad 3 \; \mu < \nu \tag{52}$$

for the $4n$ quantities $X_{j\mu}$, $j = 1, \ldots n$, $\mu = 0, 1, 2, 3$ has the rank

N	1	2	$\geqslant 3$
rank	3	5	6

Proof.

If R is a non-singular linear transformation of four dimensional space, it is clear that the set of equations (52) has the same rank as the set

$$\sum_{j=1}^{n} [(Rz_j)_\mu (RX_j)_\nu - (Rz_j)_\nu (RX_j)_\mu] = 0, \quad \mu_1 \nu = 0, 1, 2, 3$$

for the $4n$ quantities $(RX_j)_\mu$. Then, with a suitable choice of R, the last $(4 - N)$ components of vectors Rz_j, $j = 1, \ldots n$ can be made to vanish. Having reduced the problem to this simplified form, we drop the R and assume the last $4 - N$ of the components of the Z_j vanish.

To find the number of linearly independent equations (52), consider a possible linear dependence among them

$$\sum_{\substack{\mu,j=0\\ \mu<\nu}}^{3} a^{\mu\nu} \sum_{j=1}^{n} (z_{j\mu}X_{j\nu} - z_{j\nu}X_{j\mu}) = 0$$

for all $X_{l,k}$, i.e.,

$$\sum_{\substack{\mu=0\\ \mu<k}}^{3} a^{\mu\varkappa} z_{l\mu} - \sum_{\substack{\nu=0\\ k<\nu}}^{3} a^{\varkappa\nu} z_{l\nu} = 0\,,\ l = 1, \ldots n,\ \varkappa = 0, 1, 2, 3$$

or

$$Az_l = 0\,,\ l = 1, \ldots n,$$

where we have written

$$-A^{\nu\mu} = A^{\mu\nu} = a^{\mu\nu},\ \mu<\nu,\ A^{\mu\mu} = 0\,,\ \mu, \nu = 0, 1, 2, 3.$$

Thus, the rank of the equations (52) is six minus the number of linearly independent skew symmetric matrices, A such that $Az_i = 0\,,\ i = 1, 2, \ldots n$. The most general A has the form $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ where A' is an arbitrary anti-symmetric $(4-N)\times(4-N)$ matrix. The numbers tabulated in the Lemma are just six minus the number of linearly independent A'.

Lemma 7.

Let $z_1, \ldots z_n$ be a set of n four-vectors of which some four-element subset (or n element subset if $n<4$) is linearly independent. Then, the number of linearly independent solutions of equations (52) which arise from scalar products $f^{(i)}$ of the four vectors according to

$$X_{j\nu}^{(i)} = \frac{\partial f^{(i)}}{\partial z_{j\nu}}$$

is $4\,n-6$ if $n \geqslant 3$, is 3 if $n = 2$, and 1 if $n = 1$.

Proof.

It is clear that, for $n>3$, *not more* than $4\,n-6$ linearly independent solutions of equations (52) can be obtained from scalar products because there are at most $4\,n-6$ functionally independent scalar products. This follows immediately from equation (34), which expresses $z_k \cdot z_l$, k, $l \geqslant 5$ in terms of $z_i \cdot z_j$, where $i = 1, 2, \ldots n, j = 1, 2, 3, 4$. Of these $4\,n$, 6, namely $z_i \cdot z_j$, $i>j$, i $= 1, 2, 3, 4$ are expressible in terms of the rest. For $n = 3, 2, 1$ it is obvious that there are respectively 6, 3 and 1 independent scalar products at most and therefore 6, 3, and 1 linearly independent solutions of (52) at most.

To show that these upper limits on the number of solutions are actually realized under the hypothesis of the Lemma, we proceed as follows.

Denote a vector in the space of solutions of (52) by $(\zeta_1, \ldots \zeta_n)$ where the ζ_i are four-dimensional vectors. This notation is chosen so that an invariant analytic function F generates a solution $\left(\frac{\partial F}{\partial z_1} \cdots \frac{\partial F}{\partial z_n}\right)$ where $\frac{\partial F}{\partial z_j}$ stands for the four-dimensional vector with components, $\frac{\partial F}{\partial z_j^{\mu}}$. In this notation, the solution of (52) which comes from the scalar product $z_j \cdot z_k$ is

$$\underset{\xrightarrow[\qquad k^{th}\ place \qquad]{}}{(\underset{\xrightarrow[j^{th}\ place]{}}{0, \ldots 0,\ z_k},\ 0, \ldots 0,\ z_j},\ 0, \ldots 0), \tag{53}$$

and from z_j^2

$$(\underset{\xrightarrow[j^{th}\ place]{}}{0, \ldots 0,\ 2\, z_j},\ 0, \ldots 0), \tag{54}$$

By convention, we will accept (54) as the value of (53) for $j = k$.

The most general solution of (52) arising from scalar products is of the form

$$(\zeta_1, \ldots \zeta_n) = \sum_{j,k=1}^{n} a_{jk}\,(0, \ldots 0,\ z_k,\ 0, \ldots 0,\ z_j,\ 0, \ldots 0).$$

Here, evidently

$$\zeta_j = \sum_{k=1}^{n} (a_{jk} + a_{kj})\, z_k,$$

so that the antisymmetric part of the matrix a_{ik} does not contribute and a_{jk} may as well be chosen symmetric.

Now, for convenience, let the first four z_j (or first n if $n < 4$) be the linearly independent set whose existence is assumed in the Lemma. Then we can write

$$z_k = \sum_{l=1}^{m} b_{kl} z_l,\ k = 1, \ldots n, \tag{55}$$

where $m = \min(4, n)$ and b is an $n \times m$ matrix of rank m. Substituting (55) into the expression for the solution vector we see that, in the basis for solution vectors provided by $(z_{l_1}, z_{l_2}, \ldots z_{l_n})$, $l_j = 1, \ldots m$, $j = 1, \ldots n$, the most general solution arising from scalar products is of the form of an $n \times m$ matrix AB, where A is an arbitrary symmetric $n \times n$ matrix and B is a fixed $n \times m$ matrix of rank m.

To count the number of linearly independent AB simply, write B as the product of a non-singular $n \times n$ matrix B' and the $n \times m$ matrix whose first m rows form the unit matrix and whose last $n - m$ are zero:

$$B = B' \begin{pmatrix} 1 \\ \hline 0 \end{pmatrix}.$$

This is always possible because B is of rank m. The number of linearly independent matrices AB is the same as the number of linearly independent matrices

$$(B')^{\mathsf{T}} AB' \begin{pmatrix} 1 \\ \hline 0 \end{pmatrix},$$

i.e., the same as the number of linearly independent matrices of the form

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix},$$

where S_2 is an arbitrary $(n-m)\times m$ matrix and S_1 is an arbitrary symmetric $m\times m$ matrix. There are obviously $m(n-m)+\frac{1}{2}\,m(m+1)$ linearly independent of these, which immediately yields the statement of the Lemma.

Completion of the proof.

Lemmas 1 to 7 establish the single valuedness, boundedness, and continuity of f everywhere on $\mathfrak{M}_n$, and its analyticity on $\mathfrak{M}_n$ at every non-exceptional point, i.e., every point where the matrix $z_i \cdot z_j$, $i,j = 1, 2 \ldots n$, has the maximum possible rank, min $(4,n)$. To complete the proof of the theorem, we want to show that in those cases where the set of exceptional points is not singular in the sense of algebraic geometry, viz. $n = 1, 2, 3, 4$, f is also analytic there. For this purpose, we use a standard theorem on removable singularities which asserts*: Let f be a function which is analytic in a neighbourhood of a point, P, with the possible exception of a variety passing through P, the variety being defined as the set of zeros of a function analytic in the neighbourhood of P. Suppose that f is continuous or merely bounded throughout the neighbourhood of P. Then f is analytic throughout the neighbourhood of P. In our case, the variety is obtained by setting the analytic function det $(z_j \cdot z_k) = 0$. The required analyticity and continuity of f having been established by our Lemmas 1–7, the proof of the theorem is complete.

2. The varieties $\mathfrak{M}_n$.

As we have seen in Lemma 3, every rank $\leqslant 4$ complex symmetric matrix is a matrix of scalar products of four vectors, so that $\mathfrak{M}_n$ is a subset of the set of all complex symmetric $n\times n$ matrices of rank $\leqslant 4$. The same Lemma

* B and M, p. 173, theorem 5.

shows that $\mathfrak{M}_n$ is an open subset. It is clearly connected because it is the continuous image of a connected set, the tube. It is also simply connected by virtue of Lemma 3, although we shall forgo a formal proof. (The idea is, given a closed curve on $\mathfrak{M}_n$ which has to be shrunk to a point, to construct a closed curve of vectors in the tube whose image in $\mathfrak{M}_n$ is the given curve. Then, because the tube is simply connected, the curve of vectors can be shrunk to a point which implies that their image can be shrunk to a point.) Not every rank < 4 complex symmetric $n \times n$ matrix, Z, is in $\mathfrak{M}_n$, for example if Z has real positive diagonal elements it is not in $\mathfrak{M}_n$. We shall not attempt a quantitative characterization of $\mathfrak{M}_n$ at this stage, but only remark that it need not be the natural domain of analyticity for the analytic functions which occur in field theory. For example, the first named author showed in his thesis[9] that the local commutativity conditions, I equation (11), always make it possible to extend the analytic function determined by the three-fold vacuum expectation value $(\Psi_0\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \Psi_0)$ beyond $\mathfrak{M}_2$. On the other hand, it is clear that such functions cannot in general be extended to all complex symmetric rank $\leqslant 4$, $n \times n$ matrices because they must have branch lines in order to conform with physical requirements. (See, for example, the discussion of $F^{(2)}(z_{12}^2)$ in I, Section 4.)

The restriction to $n \times n$ matrices of rank $\leqslant 4$ is of course no restriction at all for $n \leqslant 4$ so the $\mathfrak{M}_n$ for $n \leqslant 4$ are open sets in Euclidean $1/2\, n\,(n+1)$ space. For $n \leqslant 5$ the restriction to rank $\leqslant 4$ on an $n \times n$ matrix Z_{ij} can be stated

$$\det \begin{pmatrix} Z_{i_1 j_1} & \cdots & Z_{i_1 j_5} \\ \\ Z_{i_5 j_1} & \cdots & Z_{i_5 j_5} \end{pmatrix} = 0$$

for each pair of five element subsets $i_1, \ldots i_5$ and $j_1, \ldots j_5$ of $1, 2, \ldots n$. The tangent spaces of $\mathfrak{M}_n$ are determined by taking the differential of the left hand side of (56). The result is a set of linear equations for the dZ_{ij} whose coefficients are determinants of 4×4 minors of Z. At any point of $\mathfrak{M}_n$ where all determinants of 4×4 principal minors of Z vanish, these equations are satisfied for any choice of dZ_{ij}. Consequently, at such a singular point, the tangent space is $1/2\, n\,(n+1)$ dimensional. On the other hand, as we learned in Lemma 7, the dimension at a non-singular point is $4\,n-6$.

Now we want to study the relation of the points of $\mathfrak{M}_n$ to those of the form $Z_{ij} = \xi_i \cdot \xi_j$, ξ_i real, $i,j = 1, \ldots n$. We will refer to such Z as physical because the arguments of the physically given vacuum expectation values are real vectors. We remark that every physical point Z is either in the interior of $\mathfrak{M}_n$ or on its boundary, because $\xi_j \cdot \xi_k$ is the limit of $(\xi_j - i\eta_j) \cdot (\xi_k - i\eta_k)$ as the η's approach zero. No physical point Z can be in the interior of $\mathfrak{M}_n$ if any of the vectors $\xi_j, j = 1, 2, \ldots, n$ (of which Z is the set of scalar products) is light-like or time-like. To see this, consider a general point $\xi'_j - i\eta_j$, $j = 1, 2, \ldots n$ of the tube. The corresponding point of $\mathfrak{M}_n$ is given by

$$Z'_{jk} = \xi'_j \cdot \xi'_k - \eta_j \cdot \eta_k - i(\xi'_j \cdot \eta_k + \eta_j \cdot \xi'_k).$$

If Z' is to be real, it is necessary that each of the ξ'_j be space-like since it is orthogonal to a vector inside the light cone. But then the diagonal elements of Z', $\xi'^2_j - \eta^2_j, j = 1, \ldots n$, are negative so that Z' can be a physical point $Z, Z_{jk} = \xi_j \cdot \xi_k, j,\varkappa = 1, \ldots n$ only if the vectors ξ_j satisfy $\xi^2_j = \xi'^2_j - \eta^2_j$. On the other hand, as we now will show, some physical points with space-like ξ_j do lie in the interior of $\mathfrak{M}_n$. Since this is a fact of considerable physical significance, and the geometrical relationships are rather involved, it is worth introducing some notation to describe the situation. We denote by S_n the set of all physical points, Z, which arise from space like vectors, i.e. of the form $Z_{jk} = \xi_j \cdot \xi_k, j, k = 1, \ldots n$ with ξ_j real and space-like. The subset of S_n which arises from $\xi_j, j = 1, \ldots n$, lying in a space-like three dimensional linear manifold, we will denote by T_n. We will also call T_n the equal time-manifold since it is the set of matrices whose elements can be taken as scalar products of vectors arising from vectors $\xi_j = x_j - x_{j+1}$, $j = 1, \ldots n$, where the x_j have equal first components.

We first prove that a subset of T_n lies in the interior of $\mathfrak{M}_n$, and then pass to neighbourhoods of that subset. Consider the vectors $\xi'_j - i\eta_j$, $j = 1, \ldots n$, where $\eta_j = \alpha_j \eta$, α_j is a real positive number, η is a real unit vector in the direction of the time axis, and ξ'_j, $j = 1, \ldots n$ are real vectors with zero component in the time direction and in one space direction, say the direction of the third axis. Then,

$$Z' = (\xi'_j - i\eta_j) \cdot (\xi'_k - i\eta_k) = \xi'_j \cdot \xi'_k - \alpha_j \alpha_k = \xi_j \cdot \xi_k,$$

where ξ_j is defined as ξ'_j plus a vector along the third axis with component α_j. The vectors ξ_j evidently all have zero time components, so that Z' is in T_n. Although the point $\xi_1, \ldots \xi_n$ does not lie in the tube, it must, by Lemmas 2 and 3, lie in the extended tube. Furthermore, by suitable choice of the components of ξ'_j, $j = 1, \ldots n$, it can be arranged that Z' has rank three. By Lemma 3, it then follows that vectors lying in neighbourhoods of ξ_j have scalar products which cover full neighbourhoods of Z' in $\mathfrak{M}_n$. Thus, the fact that the particular points Z' chosen above lie in $\mathfrak{M}_n$ implies that the physical points which arise from all $\xi_1, \ldots \xi_n$ lying in a suitably small neighbourhood of the chosen $\xi_1, \ldots \xi_n$ also lie in $\mathfrak{M}_n$. This shows that $T_n \cap \mathfrak{M}_n$ and $S_n \cap \mathfrak{M}_n$ have the same dimension as T_n and S_n, respectively.

Not all of S_n lies in $\mathfrak{M}_n$, but we will not attempt to prove this now nor to characterize those points of S_n which lie on the boundary of $\mathfrak{M}_n$. We content ourselves here with the simplest consequences of the preceding results on T_n and S_n. The set $S_n \cap \mathfrak{M}_n$ has been shown to be of the same dimension as S_n and to contain, for suitably chosen ε, all real symmetric matrices Z' of rank $\leqslant 4$ satisfying $|Z'_{jk} - \xi_j \cdot \xi_k| < \varepsilon$. This set is a real environment* for an analytic function defined on $\mathfrak{M}_n$; an analytic function f is uniquely determined all over $\mathfrak{M}_n$ if its values are given on this set. We shall see in the next section that this result has important physical consequences.

The points of T_n are always of rank $\leqslant 3$ so that for $n \geqslant 5$, $T_n \cap \mathfrak{M}_n$ lies in the singular subset of $\mathfrak{M}_n$ where the ordinary definition of analyticity fails. The set $T_4 \cap \mathfrak{M}_4$ is of (real) dimension nine while $S_4 \cap \mathfrak{M}_4$ is of dimension ten so that $T_4 \cap \mathfrak{M}_4$ is not a real environment. On the other hand, $T_1 \cap \mathfrak{M}_1$, $T_2 \cap \mathfrak{M}_2$ and $T_3 \cap \mathfrak{M}_3$ have the same dimension as $S_1 \cap \mathfrak{M}_1$, $S_2 \cap \mathfrak{M}_2$ and $S_3 \cap \mathfrak{M}_3$, so that they form real environments for analytic functions on $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{M}_3$, respectively.

3. Physical Applications.

Some physical applications of the theorem of Section 1 were already discussed in I (See, for example, the formulation of local commutativity given in I equation (11).) They arise, like those to be discussed below, when the theorem is applied to the invariant analytic function

$$F^{(n)}(z_1, z_2, \ldots z_{n-1})$$

whose boundary value, as all $\eta_j \to 0$, $j = 1, 2, \ldots n$, is the vacuum expectation value

$$F^{(n)}(\xi_1, \ldots \xi_{n-1}) = (\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0).$$

Here, in a notation somewhat different from I, we have written

$$z_j = \xi_j - i\eta_j \text{ and } \xi_j = x_j - x_{j+1} \quad j = 1, \ldots n-1.$$

The first consequence of the theorem is that $F^{(n)}(\xi_1, \ldots \xi_{n-1})$ *is an analytic function of the real variables* $Z_{ij} = \xi_i \cdot \xi_j$ $i, j = 1, \ldots n-1$ *in an open subset of the set where all* ξ_j *are space-like,* i.e. *in the notation of the preceding section, as long as* Z *belongs to a certain open subset of* $S_{n-1} \cap \mathfrak{M}_{n-1}$. This con-

* See B and M, pp. 33–34, for the definition of a real environment in Euclidean space. The same definition works here because a neighbourhood of a non-singular point in $\mathfrak{M}_n$ is essentially a Euclidean neighbourhood.

clusion follows immediately from the analysis which showed that $S_{n-1} \cap \mathfrak{M}_{n-1}$ is a real environment in $\mathfrak{M}_{n-1}$. (Recall that $\mathfrak{M}_n$ is the set of all complex symmetric matrices of the form $z_i \cdot z_j$, $i,j = 1, 2, \ldots n$, with z_i in the tube. Of course, on $\mathfrak{M}_n$ for $n \geqslant 5$, the ordinary definition of analyticity has no meaning at the exceptional points where the rank of $z_i \cdot z_j$ is less than 4. See the discussion in the outline of the proof in Section 1.

Furthermore, *the vacuum expectation value*

$$(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0)$$

is uniquely determined from its values for space-like separated $x_1, \ldots x_n$. It is possible to regard this result as a quantitative formulation of the intuitive feeling that in a Lorentz invariant theory the equivalence of descriptions in different Lorentz frames should somehow restrict the possible correlations between the values of physical quantities at different points in space time.

For $F^{(2)}(\xi_1)$, $F^{(3)}(\xi_1, \xi_2)$, and $F^{(4)}(\xi_1, \xi_2, \xi_3)$ an even more striking result holds:

$$(\Psi_0, \varphi(x_1)\varphi(x_2)\Psi_0), \quad (\Psi_0, \varphi(x_1)\varphi(x_2)\varphi(x_3)\Psi_0),$$

and

$$(\Psi_0, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\Psi_0)$$

are uniquely determined from their values at equal times, i.e., in the notation of the preceding section from their values for $Z_{ij} = \xi_i \cdot \xi_j$ *with* $Z \varepsilon T_1$, T_2, T_3, *respectively*. We want to emphasize that all three of these results hold in both local and non-local field theory.

The most important application of the preceding remarks we know of is to the proof of the following theorems which are extensions of results stated by R. Haag[10].

Theorem (Generalized Haag's Theorem First Part).

Let two theories (distinguished by a subscript $j = 1, 2$) of a neutral scalar field be given whose canonical variables are related at time t by a unitary transformation, V:

$$U_j(\vec{a}, R)\varphi_j(\vec{x}, t)U_j(\vec{a}, R)^{-1} = \varphi_j(R\vec{x} + \vec{a}, t) \quad j = 1, 2 \tag{57}$$

$$U_j(\vec{a}, R)\pi_j(\vec{x}, t)U_j(\vec{a}, R)^{-1} = \pi_j(R\vec{x} + \vec{a}, t) \quad j = 1, 2. \tag{58}$$

(Transformation law of field variable and canonical conjugate under Euclidean transformation.)

$$\left.\begin{aligned} &[\pi_j(\vec{x}, t), \varphi_j(\vec{y}, t)] = i^{-1}\delta(\vec{x} - \vec{y}) \quad j = 1, 2 \\ &[\pi_j(\vec{x}, t), \pi_j(\vec{y}, t)] = 0, \ [\varphi_j(\vec{x}, t), \varphi_j(\vec{y}, t)] = 0 \end{aligned}\right\} \tag{59}$$

(Commutation Relations)

$$\varphi_2(\vec{x}, t) = V\varphi_1(\vec{x}, t)V^{-1}, \quad \pi_2(\vec{x}, t) = V\pi_1(\vec{x}, t)V^{-1}. \tag{60}$$

Here $(\vec{a}, R)$ represents the Euclidean transformation: rotate by R and translate by $\vec{a}$; the unitary transformations, $U_j(\vec{a}, R)$, give the corresponding transformations of the states in the two theories.

Then

$$U_2(\vec{a}, R) = VU_1(\vec{a}, R)V^{-1}. \tag{61}$$

If each of the theories contains a unique normalizable state Ψ_{0j}, $j = 1, 2$, invariant under Euclidean transformation:

$$U_j(\vec{a}, R)\Psi_{0j} = \Psi_{0j} \quad j = 1, 2,$$

then

$$c\Psi_{02} = V\Psi_{01},$$

where c is a constant of absolute value 1.

Proof.

From (57), (58), and (60) we can easily derive that the operators

$$U_1(\vec{a}, R)^{-1}V^{-1}U_2(\vec{a}, R)V \tag{62}$$

commute with $\varphi_1(\vec{x}, t)$ and $\pi_1(\vec{x}, t)$ for all $\vec{x}$. Because the φ_1 and π_1 form an irreducible set* of operators, (62) must be a constant multiple of the identity operator: $\omega(\vec{a}, R)\,1$ and

$$U_2(\vec{a}, R) = \omega(\vec{a}, R)VU_1(\vec{a}, R)V^{-1}. \tag{63}$$

From (63), it follows that $(\vec{a}, R) \to \omega(\vec{a}, R)$ is a continuous unitary one dimensional representation of the Euclidean group and therefore $\omega(\vec{a}, R) = 1$.[11] This completes the proof of the first half of the theorem. To prove the second half, note that

$$U_1(\vec{a}, R)\Psi_{01} = \Psi_{01}$$

and (63) imply

$$U_2(\vec{a}, R)V\Psi_{01} = V\Psi_{01}.$$

Thus, by the uniqueness of Ψ_{02}, $V\Psi_{01}$ is a multiple, c, of Ψ_{02}. Since V is unitary, $|c| = 1$.

* That $\varphi_1(\vec{x}, t)$ and $\pi_1(\vec{x}, t)$ form an irreducible set is what we mean by our assumption that the theory is a theory of the scalar field φ_1. This assumption is made for simplicity. In a theory in which the field φ_1 interacted with a spinor field, ψ, one would only have to introduce the hypothesis that $\varphi_1, \pi_1, \psi, \bar{\psi}$, form an irreducible set, together with the appropriate extension of (57) . . . (61), to obtain an analogous theorem.

It should be noticed that only the properties of the fields and states under Euclidean transformation at time t have been used in the proof of the theorem. To one accustomed to the formalism of non-relativistic quantum mechanics, the conclusion of the theorem is in no way surprising; V always exists in such theories. Of course, there, V is a function of time, and physically different theories will give a different time dependence for V. The surprise comes when, following HAAG, one combines the preceding assumptions with those of relativistic invariance.

Theorem (Generalized HAAG's Theorem Part II).

Let two theories of a neutral scalar field be given satisfying the hypotheses of the preceding theorem. Let the theories be invariant under inhomogeneous Lorentz transformations (a, Λ) and suppose the fields transform as follows:

$$U_j(a,\Lambda)\,\varphi_j(x)\,U_j(a,\Lambda)^{-1} = \varphi_j(\Lambda x + a) \qquad j = 1, 2. \tag{64}$$

Suppose further that the states Ψ_{0j} are invariant under inhomogeneous Lorentz transformation

$$U_j(a, \Lambda)\,\Psi_{0j} = \Psi_{0j} \qquad j = 1, 2, \tag{65}$$

and that no states of negative energy exist.

Then the first four vacuum expectation values are equal in the two theories.

$$(\Psi_{01}, \varphi_1(x_1) \ldots \varphi_1(x_n)\,\Psi_{01}) = (\Psi_{02}, \varphi_2(x_1) \ldots \varphi_2(x_n)\,\Psi_{02}) \tag{66}$$

Proof.

From the preceding theorem we have for equal times $x_1^0 = x_2^0 = \ldots = x_n^0$:

$$\begin{aligned}(\Psi_{01}, \varphi_1(x_1) \ldots \varphi_1(x_n)\,\Psi_{01}) &= (V\Psi_{01}, V\varphi_1(x_1)V \ldots .V\varphi_1(x_n)V^{-1}V\Psi_{01}) \\ &= (\Psi_{02}, \varphi_2(x_1) \ldots \varphi_2(x_n)\,\Psi_{02}).\end{aligned}$$

Thus, all vacuum expectation values are equal for equal times in the two theories. For $n = 1, 2, 3, 4$, equality for all times $x_1^0 \neq x_2^0 \neq \ldots \neq x_n^0$ follows from equality for equal times by the argument presented earlier in this section. This completes the proof. The hypotheses about the absence of negative energy states and the existence of the vacuum are necessary in order that the vacuum expectation values be boundary values of analytic functions to which our previous analysis applies.

It should be noticed that we have *not* made the assumption that the two theories transform according to equivalent representations of the inhomogeneous Lorentz group; our hypotheses do not exclude a priori the possibility that the two theories have different bound states, for example. Further, we have not assumed any particular transformation law for the operators π under Lorentz transformations and time translations. Only the behaviour of $\pi(\vec{x}, t)$ for one particular time under Euclidean transformations is needed.

The uniqueness of the vacuum state is crucial to the argument. If it were possible to form normalizable states of zero three-momentum from states of mass greater than zero, the hypothesis of a unique normalizable state of zero three-momentum would be unnatural and the second theorem physically trivial. V could then carry Ψ_{01} into a superposition of Ψ_{02} and those other states of zero three-momentum. However, Wigner's analysis of the unitary representations of the inhomogeneous Lorentz group[12] shows that states can never be normalizable which are superpositions of states of mass greater than zero and have zero three-momentum, and Haag's theorem is very far from physically trivial.

As a particular case one can take the field φ_1 to be a free field satisfying

$$(\Box + m^2)\, \varphi_1(x) = 0\,, \ [\varphi_1(x), \varphi_1(y)] = i^{-1} \Delta(x-y).$$

Then we conclude: *no theory of interaction exists in which the ordinary representation of the annihilation and creation operators is used and the first four vacuum expectation values differ from their free field values. If relativistic theories of interaction exist with vacuum expectation values, $F^{(n)}$, different from the free field values for $n = 1, 2, 3, 4$, either they must use other representations of the canonical commutation relations or they do not satisfy the canonical commutation relations at all.* (This is essentially Haag's conclusion[10].) This result shows that the situation which was found by Wightman and Schweber[13] in a special non-relativistic example is typical of relativistic theories of interaction which satisfy the canonical commutation relations (if such exist at all): For each different value of the coupling constant one must use an inequivalent representation of the commutation relations (assuming that different values of the coupling constant will give rise to some difference in the vacuum expectation values $F^{(n)}$ for $n = 1, 2, 3, 4$.) Of course, the converse is not true; inequivalent representations of the commutation relations need not always give rise to physically distinct theories.

From both the aesthetic and physical point of view, the version of the generalized Haag's theorem proved here is somewhat deficient because it only asserts the equality of the first four vacuum expectation values. It seems physically plausible that two theories in which the two-particle propagator, the vertex part, and the two-particle scattering for all energies are identical (as they must be if the first four vacuum expectation values are identical) should be completely identical. On this basis, one would conjecture the aesthetically more satisfying result that *all* vacuum expectation values coincide, which would (from the work of I Section 5) indeed imply the physical equivalence of the two theories. To prove this result along the lines of the present paper would require one to establish a unique analytic continuation out of the equal time-manifold T_n into $\mathfrak{M}_n$; it would require an analysis going essentially beyond what we have presented in Section 1. (In fact, it is not difficult to see that under the hypothesis of Section 1, the analytic continuation is *not* unique for $n \geqslant 4$.)

A second matter which the present paper leaves untouched is the question of the existence of theories which use representations of the commutation relations different from those of a free field. If it turns out that no such representation gives rise to a relativistically invariant theory which is physically interesting, that would be very strong evidence of the incompatibility of the canonical commutation relations, relativistic invariance and interaction. In fact, it would show that the fact that a field strength renormalization constant is infinite in quantum electrodynamics[14] is not a special consequence of the Hamiltonian of the theory, but a general result arising from relativistic invariance.

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1. A. S. Wightman, Phys. Rev. **101**, 860 (1956). This paper will be referred to as I. We adopt the notation of I, except when it is explicitly stated otherwise.
2. See, for example, S. Lefshetz, Algebraic Geometry, Princeton Press (1953), pp. 35–40.
3. S. Bochner and W. T. Martin, Annals of Math. **57**, 490 (1953).
4. For proof of these statements, see L. Pontrjagin, Topological Groups, Princeton Press (1939), Chapters VI and IX. See especially pp. 265–66.
5. For a proof of this basic property of analytic functions, see S. Bochner and W. T. Martin, Several Complex Variables, Princeton Press (1948), p. 34, theorem 4. We shall refer to this book as B and M.
6. See, for example, L. Kronecker, Journal für Reine u. Angew. Math. **72**, 152 (1870) or C. C. Mac Duffee, Ergebnisse der Math. **2**, 364 (1933).
7. The constructions of these paragraphs are given in detail, because they solve a problem in linear algebra not standard in the physical literature. They are well known in the mathematical literature. See, for example, E. Witt, Journal für Math. **176**, 31 (1936) or C. Chevalley, The Algebraic Theory of Spinors, Columbia University Press, New York (1954), pp. 8–15. The essential points are the following. Even for a complex valued scalar product such as $z\cdot w = z_\mu w^\mu$, provided it is non-degenerate (i. e. provided $z\cdot w = 0$ for all vectors w implies $z = 0$), it remains true that a linear manifold and its orthogonal manifold $M^\perp$ have dimensions satisfying

$$\dim M + \dim M^\perp = 4.$$

However, unlike the case with a positive scalar product, it can happen that M and $M^\perp$ have non-zero vectors in common; then M and $M^\perp$ do not together span the whole space. A necessary and sufficient condition that $M \cap M^\perp = (0)$ is that M have a basis whose Gram determinant is not zero. Then, all bases for M, and $M^\perp$ as well, have non-vanishing Gram determinant. In that case, a general vector has a unique decomposition into a sum of a vector in M and a vector in $M^\perp$. There is a simple formula for the projection of a vector v into a subspace with basis $v_1, \ldots, v_m$

$$-\,|\,G(v_1 \ldots v_m)\,|^{-1} \det \left(\begin{array}{c|c} & v_1\cdot v \\ & \cdot \\ v_i\cdot v_j & \cdot \\ & \cdot \\ & v_m\cdot v \\ \hline v_1, \ldots v_m & 0 \end{array}\right)$$

If a subspace has the property that every vector in it is of zero length, it is called isotropic. Isotropic subspaces can have dimensions no larger than half the dimension of the whole space. Most of these results follow directly from arguments of the type given in the preceding paragraphs of the proof. They will be used without comment in the remainder of the proof.

8. See, for example, C. C. MacDuffee, Ref. 6, p. 408, theorem 34.1.
9. D. Hall, thesis, Princeton (1956), Chapter III, unpublished.
10. R. Haag, Mat. Fys. Medd. Dan. Vid. Selsk. **29**, no. 12 (1955), especially pp. 30–32. In the opinion of the present authors, Haag's proof is, at least in part, inconclusive. We thank Dr. Haag for a correspondence on the matter. It will not escape the discerning reader of Haag's paper that, while we have generalized his results, eliminated one of his assumptions (the asymptotic condition), completed his proofs, and sharpened his conclusions, the essential physical points are Haag's.
11. This result follows from a determination of all representations of the Euclidean group implicit in theorems of G. Mackey, Annals of Math. **55**, 101 (1951). It is also not difficult to construct a simple direct proof.
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